PERSISTENCE AND PERMANENCE OF DELAY DIFFERENTIAL EQUATIONS IN BIOMATHEMATICS

PhD Thesis

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Tartalmi kivonat

A dolgozatban nemlineáris késleltetett argumentumú skaláris differenciálegyenletek illetve differenciálegyenlet-rendszerek egy széles osztályát vizsgáljuk. Az ilyen egyenletek gyakran megjelennek természettudományi, közgazdaságtani, mérnöki, populációdinamikai, epidemiológiai alkalmazásokban. Mivel az általunk tekintett modelleket populációdinamikai alkalmazások motiválták, pozitív megoldásokra fókuszálunk, és a modellek pozitív megoldásai perzisztenciáját és egyenletes permanenciáját vizsgáljuk. A fő eredményeink alkalmazásaként explicit becsléseket fogalmazunk meg a megoldások limesz inferiorjára és limesz szuperiorjára. Egyszerű skaláris modellek esetén visszakapjuk az irodalomból ismert becsléseket, de gyengébb feltételek mellett. A bizonyításaink összehasonlító tételeken és a monoton iterációs technikán alapulnak. Rendszerek esetében a becsléseinkhez meg kell oldani egy kapcsolódó nemlineáris algebrai egyenletrendszert. Elegendő feltételeket adunk meg ilyen egyenletrendszer megoldásai létezésére és egyértelműségére. Az eredményeink ismert eredményeket terjesztenek ki lényegesen általánosabb egyenletosztályokra, és a használt feltételeink gyengébbek az irodalomban eddig vizsgált esetekhez képest. Elegendő feltételeket adunk meg bizonyos egyenletek esetében a megoldások aszimptotikus ekvivalenciájára. Az új eredményeket számos példa és numerikus szimuláció illusztrálja.

Abstract

In this work, we study a large family of scalar differential equations and systems of differential equations with delays. Such equations appear frequently as mathematical models in natural sciences, economics and engineering, population dynamics, mathematical epidemiology and other engineering applications. Since our model equations are motivated by applications in population dynamics, we focus only on positive solutions, and we investigate persistence and permanence of the positive solutions of our model equations. As an application of the main results, we obtain explicit estimates for the limit inferior and limit superior of the solutions. For some simple scalar population models, our method recovers known estimates of the literature, but under weaker conditions. Our method uses comparison technique and iterative methods of differential equations. For the system case, our results requires the solutions of an associated system of nonlinear algebraic equations. We establish sufficient conditions implying the existence and uniqueness of solutions of such system of algebraic equations. These results generalize known methods for much larger classes of equations, and our conditions are weaker for the previously studied cases too. For a special class of differential equations, we give sufficient conditions for the asymptotic equivalence of the positive solutions. All the new results are illustrated by several special examples and numerical experiments too.

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Chapter 1

Introduction

In modelling in the biological, physical and social sciences, it is sometimes necessary to take account of time delays inherent in the phenomena. In all these fields scientists need their models to behave more like the real process. Many processes include aftereffect phenomena in their inner dynamics. In these cases, it may be necessary to choose between a model with discrete delays or a model with distributed delay.

1.1 Background and motivation

Time delays of one type or another have been incorporated into biological models to represent resource regeneration times, maturation periods, feeding times, reaction times, etc. by many researchers. We refer to the monographs of ([26], [27], [35], [57], [60]) for discussions of general delayed biological systems. In general, delay differential equations exhibit much more complicated dynamics than ordinary differential equations since a time delay could cause a stable equilibrium to become unstable and cause the populations to fluctuate. In this section, we shall review various delay differential equations models arising from studying single species dynamics. Let $x(t)$ denote the population size at time t; let b and d denote the birth rate and death rate, respectively, on the time interval $[t, t+\Delta t]$, where $\Delta t > 0$. Then

$$x(t+\Delta t) - x(t) = bx(t)\Delta t - dx(t)\Delta t.$$

Dividing by Δt and letting Δt approach zero, we obtain

$$\frac{dx}{dt} = bx - dx = rx, \tag{1.1.1}$$

where $r = b - d$ is the intrinsic growth rate of the population. The solution of equation (1.1.1) with an initial population $x(0) = x_0$ is given by

$$x(t) = x_0 e^{rt}. \tag{1.1.2}$$

The function (1.1.2) represents the traditional exponential growth if $r > 0$ or decay if $r < 0$ of a population. Such a population growth, due to Malthus (1798), may be valid for a short period, but it cannot go on forever. Verhulst (1836) proposed the logistic equation

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right), \tag{1.1.3}$$

where $r > 0$ is the intrinsic growth rate and $K > 0$ is the carrying capacity of the population. In model (1.1.3), when x is small the population grows as in the Malthusian model (1.1.1); when x is large the members of the species compete with each other for the limited resources. Solving (1.1.3) by separating the variables, we obtain ($x(0) = x_0$),

$$x(t) = \frac{x_0 K}{x_0 - (x_0 - K)e^{-rt}}. \tag{1.1.4}$$

If $0 < x_0 < K$, the population grows, approaching K asymptotically as $t \to \infty$. If $x_0 > K$, the population decreases, again approaching K asymptotically as $t \to \infty$. If $x_0 = K$, the population remains in time at $x = K$. In fact, $x = K$ is called the equilibrium of equation (1.1.3). Thus, the positive equilibrium $x = K$ of the logistic equation (1.1.3) attracts all the positive solutions; that is, $\lim_{t\to\infty} x(t) = K$, for solution $x(t)$ of (1.1.3) with any positive initial value $x(0) = x_0$.

In the above logistic model it is assumed that the growth rate of a population at any time t depends on the relative number of individuals at that time. In practice, the process of reproduction is not instantaneous. For example, in a Daphnia a large clutch presumably is determined not by the concentration of unconsumed food available when the eggs hatch, but by the amount of food available when the eggs were forming, some time before they pass into the broad pouch. Between this time of determination and the time of hatching many newly hatched animals may have been liberated from the brood pouches of other Daphnia in the culture, so increasing the population. Hutchinson [52] assumed egg formation to occur τ units of time before hatching and proposed the following more realistic logistic equation

$$\frac{dx(t)}{dt} = rx(t)\left(1 - \frac{x(t-\tau)}{K}\right), \tag{1.1.5}$$

where r and K have the same meaning as in the logistic equation (1.1.3), $\tau > 0$ is a constant. Equation (1.1.5) is often referred to as the Hutchinson's equation or delayed logistic equation and was introduced with the initial condition

$$x(t) = \varphi(t), \qquad -\tau \leq t \leq 0, \tag{1.1.6}$$

where, φ is continuous on $[-\tau, 0]$.

In this Thesis we focus on the study of boundedness of the positive solutions of differential equations with time delays, that appear frequently as mathematical models in natural sciences, economics and engineering, population dynamics, mathematical epidemiology, economics and large classes of engineering applications and many others. Since our model equations are motivated by applications in population dynamics, we focus only on positive solutions, and we investigate persistence and permanence of the positive solutions.

1.2 The structure and content of the Thesis

The structure of the Thesis is the following. In Chapter 1 we give a list of notations we use in the rest of the Thesis.

In Chapter 2 we give some basic background, known results and notions on the topics we will use in later chapters in our investigation.

In Chapter 3 we study the persistence and the uniform permanence of the positive solutions of the general nonlinear scalar delay differential equation

$$\dot{x}(t) = r(t)\Big(g(t, x_t) - h(x(t))\Big), \qquad t \geq 0,$$

and present sufficient conditions which guarantee the boundedness of the solution (see Theorem 3.2.4). This general form of the equation may include a single or multiple constant or time-dependent point delay functions as well as distributed delays in the positive terms. Corollary 3.3.1 immediately implies the estimates obtained in [4], but under weaker conditions. Our method is based on the well-known comparison theorem for differential equations. We give also, in Section 3.3, several particular cases and explicit estimations for the upper and lower limit of the solutions. We investigated in some special cases conditions, which imply that all solutions have the same asymptotic behavior, i.e., the difference of any two positive solutions tends to zero.

In Chapter 4 we give sufficient conditions which imply the existence and uniqueness of the positive solutions of the general nonlinear system of algebraic equations

$$\gamma_i(x_i) = \sum_{j=1}^{n} g_{ij}(x_j), \qquad 1 \leq i \leq n.$$

Our main result, Theorem 4.2.1 below, uses a monotone iterative method to prove existence of a positive solution, and an the extension of the method used in [21] to prove uniqueness under a weaker condition than that assumed in [21]. We introduce many applications and special cases of our main results and in some cases we give necessary conditions for the existence and uniqueness of the positive solutions. Also we give a counterexample which shows the importance of our conditions.

In Chapter 5 we consider the system of nonlinear delay differential equations

$$\dot{x}_i(t) = \sum_{j=1}^{n} \sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t) h_{ij}(x_j(t - \tau_{ij\ell}(t))) - r_i(t) f_i(x_i(t)) + \rho_i(t),\ t \geq 0,\ \ 1 \leq i \leq n,$$

and give sufficient conditions for the uniform permanence of the positive solutions of the system. Also in several particular cases, explicit estimates are given for the upper and lower limit of the solutions.

In Chapter 6 we summarize the new results. Also the list of publications and conference lectures of Nahed A. Mohamady related to the topic of this Thesis is given.

In Appendix A we present some technical or long proofs.

1.3 Notations

The most important notations used throughout in this Thesis are listed below in this section.

Mathematical notations

$\mathbb{R}$	the set of real numbers
$\mathbb{R}_+ := [0, \infty)$	the set of non-negative real numbers
$C(X, Y)$	the set of continuous functions mapping from X to Y
C	the set of continuous functions mapping from $[-\tau, 0]$ to $\mathbb{R}$
C_+	the set of continuous functions mapping ψ from $[-\tau, 0]$ to $\mathbb{R}_+$ with $\psi(0) > 0$
C_0	the set of continuous functions mapping ψ from $[-\tau, 0]$ to $\mathbb{R}_+$ with $\psi(t) > 0$, $-\tau \leq t \leq 0$
$\Vert \cdot \Vert_\tau$	the maximum norm of a continuous function $x : [-\tau, 0] \to \mathbb{R}^n$ defined by $\Vert x \Vert_\tau := \max_{-\tau \leq t \leq 0} \Vert x(t) \Vert$

$x_t(\theta)$	the segment function defined by $x_t(\theta) := x(t+\theta),\ \theta \in [-\tau, 0]$, where x is a function defined from $[-\tau, \infty)$ to $\mathbb{R}$, and $t \in \mathbb{R}_+$
$\dot{x} = \frac{dx}{dt}$	time derivative of x
$\underline{x}(\infty)$	$:= \liminf_{t\to\infty} x(t)$
$\overline{x}(\infty)$	$:= \limsup_{t\to\infty} x(t)$
x_i	i^{th} element of a vector x
x^T	transpose of a vector x.

Next we list the acronyms we use in the Thesis.

Acronyms

IVP	initial value problem
ODEs	ordinary differential equations
DDEs	delay differential equations
Eq	equation.

Chapter 2

Theoretical Background

In this chapter we review some concepts and known results which are used or referred to later in the Thesis.

2.1 Scalar delay differential equations

In this section we investigate a scalar delay differential equation which will be useful in the rest of the Thesis.

Consider the scalar nonlinear differential equation with general delays

$$\dot{x}(t) = f(t, x_t), \qquad t \geq t_0, \tag{2.1.1}$$

and the initial condition

$$x(t) = \varphi(t - t_0), \qquad t_0 - \tau \leq t \leq t_0, \tag{2.1.2}$$

where $\tau > 0, t_0 \geq 0$, $\varphi \in C := C([-\tau, 0], \mathbb{R})$, $f : [t_0, \infty) \times C \to \mathbb{R}$ is continuous and $x_t(\theta) := x(t + \theta)$, $\theta \in [-\tau, 0]$.

Definition 2.1.1. *A function x is called a solution of Eq. (2.1.1) on $[t_0 - \tau, \infty)$ if $x \in C([t_0 - \tau, \infty), \mathbb{R})$, (2.1.2) holds and x satisfies Eq. (2.1.1) for $t \in [t_0, \beta)$ for some $\beta > t_0$ or for $t \in [t_0, \infty)$.*

Definition 2.1.2. *A function $f : [t_0, \infty) \times C \to \mathbb{R}$ is called locally Lipschitz in its* *second variable, if for any $t \in [t_0, \infty)$ and $\varphi \in C$, there exist $\delta_1 > 0, \delta_2 > 0$ and $L > 0$ constants such that*

$$\|f(s, \psi_1) - f(s, \psi_2)\| \leq L\|\psi_1 - \psi_2\|_\tau,$$

for $s \in [t-\delta_1, t+\delta_1]$ and $\psi_1, \psi_2 \in C$ satisfying $\|\psi_1-\varphi\|_\tau \leq \delta_2$ and $\|\psi_2-\varphi\|_\tau \leq \delta_2$, where $\|\psi\|_\tau := \max_{-\tau \leq s \leq 0} \|\psi(s)\|$.

We recall, from [30], the following theorem of the existence and uniqueness of solution of the IVP (2.1.1) and (2.1.2).

Theorem 2.1.1. *[30] Let $f : \mathbb{R}_+ \times C \to \mathbb{R}$ be continuous and locally Lipschitz continuous in its second variable. Then, for every $t_0 \geq 0$ and $\varphi \in C$, there exists $\beta > t_0$ such that the IVP (2.1.1) and (2.1.2) has a unique solution on $[t_0 - \tau, \beta)$.*

We note that this result can be naturally extended to systems of delay differential equations too.

The following comparison theorem of differential equations will be essential in our proofs later.

Theorem 2.1.2. *Let $f : \mathbb{R}_+ \times C \to \mathbb{R}$ and $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be continuous, $\phi \in C$, and $t_0 \geq 0$ be fixed. Let x be a solution of the IVP*

$$\dot{x}(t) = f(t, x_t), \qquad t \geq t_0, \tag{2.1.3}$$

$$x(t) = \varphi(t - t_0), \qquad t \in [t_0 - \tau, t_0], \tag{2.1.4}$$

and let y be a unique solution of the IVP

$$\dot{y}(t) = g(t, y(t)), \qquad t \geq t_0, \tag{2.1.5}$$

$$y(t_0) = \varphi(0). \tag{2.1.6}$$

Then if $f(t, \psi) \geq g(t, \psi(0))$, for all $(t, \psi) \in \mathbb{R}_+ \times C$, there follows $x(t) \geq y(t)$ for $t \geq t_0$. Also, if $f(t, \psi) \leq g(t, \psi(0))$, for all $(t, \psi) \in \mathbb{R}_+ \times C$, there follows $x(t) \leq y(t)$ for $t \geq t_0$.

Proof. The proof is given in [15] for the case when (2.1.3) is an ODE, but it can be easily extended to this case too. □

The notions of persistence and permanence are frequently studied in mathematical biology (see e.g. [57, 60]). Following [3, 4, 31] and [33] we define the next two notions. Let us, first, define the class $C_+ := \{\psi \in C([-\tau, 0], \mathbb{R}_+) : \psi(0) > 0\}$, where $\tau > 0$.

Definition 2.1.3. *Eq. (2.1.1) is said to be persistent in C_+ if any positive solution $x(t)$ is bounded away from zero, i.e.,* $\liminf_{t\to\infty} x(t) > 0$.

Definition 2.1.4. *Eq. (2.1.1) is called uniformly permanent if there exist two positive numbers m and M with $m < M$ such that, all positive solutions $x(t)$ of Eq. (2.1.1) satisfy*

$$m \leq \liminf_{t\to\infty} x(t) \leq \limsup_{t\to\infty} x(t) \leq M.$$

2.2 Mathematical and biological models

In this section, we look at some ways mathematics is used to model dynamic processes in biology. Interactions between the mathematical and biological sciences have been appearing rapidly in recent years. Both traditional topics, such as population and disease modeling, and new ones, have made biomathematics an exciting field. Simple formulas relate, for instance, the population of a species in a certain year to that of the following year. We consider the biological models as nonlinear delay differential equations. Although many of the models we examine may at first seem to be gross simplifications, their very simplicity is a strength. Simple models show clearly the implications of our most basic assumptions. We begin by considering the scalar nonautonomous differential equation

$$\dot{N}(t) = a(t)N(t) - r(t)N^2(t), \qquad t \geq 0 \tag{2.2.1}$$

which is known as the logistic equation in mathematical ecology. Eq. (2.2.1) is a prototype in modeling the dynamics of single species population systems whose biomass or density is denoted by a function N of the time variable. The functions $a(t)$ and $r(t)$ are time dependent net birth and self-inhibition rate functions, respectively. The carrying capacity of the habitat is the time dependent function

$$K(t) = \frac{a(t)}{r(t)}, \qquad t \geq 0. \tag{2.2.2}$$

By using this notation, Eq. (2.2.1) can be written as

$$\dot{N}(t) = r(t)\Big(K(t)N(t) - N^2(t)\Big), \qquad t \geq 0, \tag{2.2.3}$$

or

$$\dot{N}(t) = r(t)\Big(K_0 N(t) - N^2(t)\Big), \qquad t \geq 0 \tag{2.2.4}$$

whenever the carrying capacity is constant, i.e., $K(t) = K_0$, $t \geq 0$ with a $K_0 > 0$.

It follows by elementary techniques that the above equations with the initial condition

$$N(0) = N_0 > 0 \tag{2.2.5}$$

has a unique solution $N(N_0)(t)$ of the initial value problem (IVP) (2.2.4) and (2.2.5) given by the explicit formula

$$N(N_0)(t) = \frac{N_0 K_0 e^{K_0 \int_0^t r(s)\,ds}}{K_0 + N_0(e^{K_0 \int_0^t r(s)\,ds} - 1)}, \qquad t \geq 0. \tag{2.2.6}$$

From the above formula, we get that either

$$\int_0^\infty r(s)\,ds = \infty \tag{2.2.7}$$

and

$$N(N_0)(\infty) := \lim_{t\to\infty} N(t) = K_0 \qquad \text{for any } N_0 > 0,$$

or

$$\int_0^\infty r(s)\,ds < \infty \tag{2.2.8}$$

and

$$N(N_0)(\infty) = \frac{N_0 K_0 e^{K_0 \int_0^\infty r(s)\,ds}}{K_0 + N_0(e^{K_0 \int_0^\infty r(s)\,ds} - 1)} \neq K_0 \qquad \text{for any } N_0 \neq K_0.$$

Thus K_0 is a global attractor of (2.2.4) with respect to the positive solutions if and only (2.2.7) holds.

It follows by some elementary technique that for any $N_0 > 0$ the solution $N(N_0)(t)$ of the IVP (2.2.3) and (2.2.5) obeys

$$\underline{K}(\infty) \leq \liminf_{t\to\infty} N(N_0)(t) \leq \limsup_{t\to\infty} N(N_0)(t) \leq \overline{K}(\infty) \tag{2.2.9}$$

for any $N_0 > 0$, if

$$0 < \underline{K}(\infty) := \liminf_{t\to\infty} K(t) \leq \limsup_{t\to\infty} K(t) =: \overline{K}(\infty) < \infty \tag{2.2.10}$$

and (2.2.7) holds.

In (1948) Hutchinson [52] considered the delayed logistic equation

$$\dot{N}(t) = rN(t)\left(1 - \frac{N(t-\tau)}{K_0}\right), \tag{2.2.11}$$

where $r = b - d$ is the intrinsic growth rate of the population and $K_0 > 0$ has the same meaning as in the logistic equation (2.2.4), $\tau > 0$ is a constant. Equation (2.2.11) was introduced with the initial condition

$$N(t) = \varphi(t), \qquad -\tau \leq t \leq 0, \tag{2.2.12}$$

where, φ is continuous on $[-\tau, 0]$. It is interesting to note that Equation (2.2.11) can be observed in some Daphnia populations. We refer the reader to ([25, 35, 47, 48, 57, 64, 65, 66, 72]) who have argued that the delay should enter in the birth term rather than in death term.

2.3 Numerical approximation of delay equations

In this section, we investigate numerical approximation of differential equations using the class of delay differential equations with piecewise constant arguments. Equations with piecewise constant arguments were introduced by Wiener [68] and

Cooke and Wiener [22, 23]. For surveys of theory and applications of such equations we refer to [1, 24, 69]. We present a numerical approximation method which was introduced first for linear delay equations in [40], and later it was extended for various classes of differential equations (see [43]).

We introduce the method for nonlinear delay equations of the form

$$\dot{x}(t) = f(t, x(t), x(t-\tau)), \quad t \geq 0, \tag{2.3.1}$$

with initial condition

$$x(t) = \varphi(t), \quad -\tau \leq t \leq 0. \tag{2.3.2}$$

Let $h > 0$ be a discretization parameter. We associate the following equation with piecewise constant arguments to the IVP (2.3.1)-(2.3.2)

$$\dot{y}_h(t) = f\left([t/h]h, y_h([t/h]h), y_h([t/h]h - [\tau/h]h)\right), \quad t \geq 0 \tag{2.3.3}$$

and

$$y_h(t) = \varphi(t), \quad -\tau \leq t \leq 0, \tag{2.3.4}$$

where $[\cdot]$ denotes the greatest integer part function.

Following [40] we have the following definition for the solution of the IVP (2.3.3)-(2.3.4):

Definition 2.3.1. *By a solution of the IVP (2.3.3)-(2.3.4), we mean a function* y_h *defined on* $\{-kh : k \in \mathbb{N}, -\tau \leq -kh \leq 0\}$ *by (2.3.4), which satisfies the following properties on* $\mathbb{R}_+$*:*

(i) the function y_h *is continuous on* $\mathbb{R}_+$*,*

(ii) the derivative $\dot{y}_h$ *exists for each* $t \in \mathbb{R}_+$ *with the possible exception of the points* $kh (k = 0, 1, 2, ...)$ *where finite one-sided derivative exist, and*

(iii) the function y_h *satisfies (2.3.3) on each interval* $[kh, (k+1)h)$ *for* $k = 0, 1, 2, ...$

The right-hand-side of (2.3.3) is constant on the intervals $[kh,(k+1)h)$, so the solution of (2.3.3)-(2.3.4) is a continuous function which is linear in between the mesh points $\{kh : k \in \mathbb{N}\}$. Define $\ell := [\tau/h]$.

We integrate both sides of (2.3.3) from kh to t,

$$\int_{kh}^{t} \dot{y}_h(s)ds = \int_{kh}^{t} f\left([s/h]h, y_h([s/h]h), y_h([s/h]h - \ell h)\right) ds,$$

where $kh \leq t < (k+1)h$. Using that the integrand on the right-hand-side is constant, we get

$$y_h(t) - y_h(kh) = f(kh, y_h(kh), y_h(kh - \ell h))(t - kh).$$

Now taking the limit $t \to (k+1)h$ from the left-hand, we have

$$y_h((k+1)h) - y_h(kh) = hf(kh, y_h(kh), y_h(kh - \ell h)).$$

Since y_h is linear between the mesh points, the values $a(k) = y_h(kh)$ uniquely determine the solution. The sequence $a(k)$ satisfies the difference equation

$$a(k+1) = a(k) + f(kh, a(k), a(k-\ell)) \cdot h, \qquad k = 0, 1, 2, \ldots,$$
$$a(-k) = \varphi(-kh), \qquad k = 0, 1, 2, \ldots, \quad -\tau \leq kh \leq 0.$$

It was shown in [40, 43] that

$$\lim_{h \to 0} |x(t) - y_h(t)| = 0, \qquad \text{for all fixed} \;\; t \geq 0.$$

In all the numerical examples of this Thesis we will use the above numerical approximation method. For other numerical methods to approximate delay equations we refer to [5].

Chapter 3

On a nonlinear scalar delay population model

In this chapter we consider a nonlinear scalar delay differential equation and establish sufficient conditions for the uniform permanence of the positive solutions of the equation.

This chapter is organized as follows: Section 3.1 introduces a description of our nonlinear delay differential equation and some basic definitions and preliminaries. Section 3.2 presents the main results of this chapter for the uniform permanence of the positive solutions of the equation. In Section 3.3, several particular cases are introduced and explicit formulas are given for the upper and lower limit of the solutions. Also, in some special cases, sufficient conditions, which imply that the difference of any two positive solutions tends to zero, are given. In Section 3.4, several examples with numerical simulations are given to illustrate the main results.

3.1 Introduction and preliminaries

In this chapter, we investigate lower and upper estimates for the positive solutions of the nonlinear scalar delay differential equation

$$\dot{x}(t) = r(t)\Big(g(t, x_t) - h(x(t))\Big), \qquad t \geq 0, \tag{3.1.1}$$

where $\tau > 0$ is fixed, $x_t(\theta) = x(t+\theta)$, $-\tau \leq \theta \leq 0$, $r, h \in C(\mathbb{R}_+, \mathbb{R}_+)$, $g \in C(\mathbb{R}_+ \times C, \mathbb{R}_+)$. Eq. (3.1.1) can be considered as a population model equation with delay in the birth term $r(t)g(t, x_t)$, and no delay in the self-inhibition term $r(t)h(x(t))$. The form of the delay model is based on the works of the authors [12, 25, 35, 47, 48, 57, 64, 65, 66, 72]. Eq. (3.1.1) includes, e.g., the next equations

$$\dot{x}(t) = \sum_{k=1}^{n} \alpha_k(t)x(t - \tau_k(t)) - \beta(t)x^2(t), \qquad t \geq 0, \tag{3.1.2}$$

$$\dot{x}(t) = \sum_{k=1}^{n} \alpha_k(t)x^p(t - \tau_k(t)) - \beta(t)x^q(t), \qquad t \geq 0, \quad 0 < p < q,\ q \geq 1, \tag{3.1.3}$$

$$\dot{x}(t) = \alpha(t)f(x(t-\tau)) - \beta(t)g(x(t)), \qquad t \geq 0, \tag{3.1.4}$$

and

$$\dot{x}(t) = \frac{\alpha(t)x(t-\tau)}{1 + \gamma(t)x(t-\tau)} - \beta(t)x^2(t), \qquad t \geq 0 \tag{3.1.5}$$

with discrete delays, or

$$\dot{x}(t) = \alpha(t)\int_{-\tau}^{0} f(s, x(t+s))\, ds - \beta(t)h(x(t)), \qquad t \geq 0 \tag{3.1.6}$$

with distributed delay.

Recently, lower and upper estimations of the positive solutions of Eq. (3.1.2) were proved in [4] and [31] under the assumptions that the coefficients α_k and β satisfy

$$\alpha_0 \leq \alpha_k(t) \leq A_0, \qquad \beta_0 \leq \beta(t) \leq B_0, \qquad t \geq 0,\ k = 1, \dots, n \tag{3.1.7}$$

with some positive constants α_0, A_0, β_0 and B_0. The following theorem, which is a consequence of our main results, illustrate that the above boundedness conditions can be released. In this statement we investigate the qualitative behavior of the solution of Eq. (3.1.2) under the initial condition

$$x(t) = \varphi(t), \qquad -\tau \le t \le 0, \tag{3.1.8}$$

where $\varphi \in C_+$. The unique solution of Eq. (3.1.2) and (3.1.8) is denoted by $x(\varphi)(t)$. We will assume

$$\alpha_k, \tau_k \in C(\mathbb{R}_+, \mathbb{R}_+),\ (k = 1, \dots, n), \qquad \tau := \max_{1\le k\le n} \sup_{t\ge 0} \tau_k(t) < \infty, \tag{3.1.9}$$

$$\beta \in C(\mathbb{R}_+, (0,\infty)), \qquad \int_0^\infty \beta(t)\, dt = \infty, \tag{3.1.10}$$

and

$$0 < \underline{m} := \liminf_{t\to\infty} \frac{1}{\beta(t)} \sum_{k=1}^n \alpha_k(t) \qquad \text{and} \qquad \overline{m} := \limsup_{t\to\infty} \frac{1}{\beta(t)} \sum_{k=1}^n \alpha_k(t) < \infty. \tag{3.1.11}$$

We note that Eq. (3.1.2) has no constant positive steady-state if the function $\frac{1}{\beta(t)} \sum_{k=1}^n \alpha_k(t)$ is not constant.

Our proof is based on using some relevant well-known theorem for differential inequalities of ordinary differential equations, moreover we can apply our method for differential equations with distributed delay, e.g., of the form (3.1.6), where techniques of [4] and [31] do not work.

3.2 Main results of Chapter 3

Throughout this chapter we use the following notations.

$$\underline{x}(\infty) := \liminf_{t\to\infty} x(t) \qquad \text{and} \qquad \overline{x}(\infty) := \limsup_{t\to\infty} x(t).$$

We consider the scalar nonlinear delay equation

$$\dot{x}(t) = r(t)\Big(g(t, x_t) - h(x(t))\Big), \qquad t \geq 0, \tag{3.2.1}$$

with the initial condition

$$x(t) = \varphi(t), \qquad -\tau \leq t \leq 0. \tag{3.2.2}$$

Next we list the following conditions, which will be used only whenever this is explicitly indicated:

($\mathbf{H_1}$) $r \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $r(t) > 0$ for $t > 0$ and $\int_0^\infty r(s)ds = \infty$, $g \in C(\mathbb{R}_+ \times C, \mathbb{R})$ with $g(t, \psi) \geq 0$ for $t \geq 0$ and $\psi(s) \geq 0$, $-\tau \leq s \leq 0$.

($\mathbf{H_2}$) $h \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies $0 = h(0) < h(x_1) < h(x_2)$ for $0 < x_1 < x_2$, and for any nonnegative constants v and L satisfying $L \neq v$ the condition $\int_L^v \frac{ds}{h(v)-h(s)} = +\infty$ holds.

($\mathbf{H_3}$) There exists $q_1 \in C(\mathbb{R}_+^2, \mathbb{R}_+)$ such that for any $T \geq 0,\ u > 0$ we have

$$g(t, \psi) \geq q_1(T, u), \quad \text{if} \quad t \geq T \text{ and } \psi \in C \text{ with } \psi(s) \geq u,\ -\tau \leq s \leq 0,$$

and there exist constants $T_1 \geq \tau$ and $u_1 > 0$ such that

$$q_1(T_1, u) > h(u), \qquad u \in (0, u_1].$$

($\mathbf{H_4}$) There exists $q_2 \in C(\mathbb{R}_+^2, \mathbb{R}_+)$ such that for any $T \geq 0,\ u > 0$ we have

$$g(t, \psi) \leq q_2(T, u), \quad \text{if} \quad t \geq T \text{ and } \psi \in C \text{ with } \psi(s) \leq u,\ -\tau \leq s \leq 0,$$

and there exist constants $T_2 \geq \tau$ and $u_2 > 0$ such that

$$q_2(T_2, u) < h(u), \qquad u \geq u_2.$$

($\mathbf{H_5}$) There exists $q_1^* \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that for any $v \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfying $\lim_{T\to\infty} v(T) = w$ we have

$$\liminf_{T\to\infty} q_1(T, v(T)) \geq q_1^*(w).$$

($\mathbf{H_6}$) There exists $q_2^* \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that for any $v \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfying $\lim_{T\to\infty} v(T) = w$ we have

$$\limsup_{T\to\infty} q_2(T, v(T)) \le q_2^*(w).$$

We note that the integral condition of $r(t)$ in ($\mathbf{H_1}$) is natural according to Section 2.2. In the proofs of our results, a comparison theorem will be used, hence we will use conditions ($\mathbf{H_3}$) and ($\mathbf{H_4}$) to estimate the birth rate function g from above and from below.

We remark that from the assumed continuity of the functions r, g, h and φ, the IVP (3.2.1) and (3.2.2) has a solution, but it is not necessary unique. Any fixed solution of (3.2.1) corresponding to the initial function φ will be denoted by $x(\varphi)(t)$, and we assume that this solution exists on $[0, \infty)$. We also note that if h is locally Lipschitz continuous, then the integral condition in ($\mathbf{H_2}$) holds.

Before we formulate our main results, we have to mention that in the proof of our main result, we compare the solutions of equation (3.2.1) with that of the associated ordinary differential equation

$$\dot{y}(t) = r(t)\Big(c - h(y(t))\Big), \qquad t \ge T \ge 0 \tag{3.2.3}$$

with the initial condition

$$y(T) = y^*, \tag{3.2.4}$$

where $c \ge 0$, and r and h satisfy ($\mathbf{H_1}$) and ($\mathbf{H_2}$). We will show in Lemma 3.2.1 below that for all $(T, y^*, c) \in (\mathbb{R}_+ \times (0, \infty) \times \mathbb{R}_+)$ the IVP (3.2.3) and (3.2.4) has a unique solution which is denoted by $y(t) = y(T, y^*, c)(t)$.

First, we prove some basic properties of the solutions of the IVP (3.2.3) and (3.2.4).

Lemma 3.2.1. *Let ($\boldsymbol{H_1}$) and ($\boldsymbol{H_2}$) be satisfied. Then for any $T \ge 0$, $y^* > 0$ and*

$c \geq 0$ the corresponding solution $y(T, y^*, c)(t)$ of the IVP (3.2.3) and (3.2.4) is uniquely defined on $[T, \infty)$, moreover we have

(i) $c > 0$ and $0 < y^* < h^{-1}(c)$ yield that

$$0 < y(T, y^*, c)(t) < h^{-1}(c), \qquad \dot{y}(T, y^*, c)(t) > 0, \quad t \geq T$$

and

$$\lim_{t\to\infty} y(T, y^*, c)(t) = h^{-1}(c);$$

(ii) $y^* = h^{-1}(c)$ yields that $y(T, y^*, c)(t) = h^{-1}(c), \quad t \geq T$;

(iii) $c \geq 0$ and $y^* > h^{-1}(c)$ yield that

$$y(T, y^*, c)(t) > h^{-1}(c), \qquad \dot{y}(T, y^*, c)(t) < 0, \quad t \geq T$$

and

$$\lim_{t\to\infty} y(T, y^*, c)(t) = h^{-1}(c).$$

Proof. See Appendix A. □

The next lemma shows that all solutions of (3.2.1) corresponding to the initial condition $\varphi \in C_+$ are positive on $[0, \infty)$.

Lemma 3.2.2. *Assume that conditions ($\boldsymbol{H_1}$) and ($\boldsymbol{H_2}$) are satisfied. Then, for any $\varphi \in C_+$, we have that $x(\varphi)(t) > 0$ for $t \in [0, \infty)$.*

Proof. Let $x(t) = x(\varphi)(t)$ be any solution of the IVP (3.2.1) and (3.2.2). Since $x(0) = \varphi(0) > 0$, there exists a $\delta > 0$ such that $x(t) > 0$ for $0 \leq t < \delta$. If $\delta = \infty$, then the proof is completed. Otherwise, there exists a $t_1 \in (0, \infty)$ such that $x(t) > 0$ for $0 \leq t < t_1$ and $x(t_1) = 0$. Since by ($\mathbf{H_1}$) $g(t, \psi) \geq 0$ for any $(t, \psi) \in [0, \infty) \times C$, from (3.2.1) we have that

$$\dot{x}(t) \geq -r(t)h(x(t)), \qquad 0 \leq t \leq t_1. \tag{3.2.5}$$

But from Theorem 2.1.2, we have

$$x(t) \geq y(t), \qquad 0 \leq t \leq t_1,$$

where $y(t) = y(0, \varphi(0), 0)(t)$ is the positive solution of (3.2.3), with $c = 0$ and with the initial condition

$$y(0) = x(0) = \varphi(0) > 0.$$

Then at $t = t_1$ we get $x(t_1) \geq y(t_1) > 0$, which is a contradiction with our assumption that $x(t_1) = 0$. Hence $x(t) > 0$ for $t \in [0, \infty)$. □

The next result implies that, under our conditions, Eq. (3.2.1) is persistent.

Lemma 3.2.3. *Let conditions (**H₁**) and (**H₂**) be satisfied. Then, for any $\varphi \in C_+$, we have*

(i) if ($\boldsymbol{H_3}$) is satisfied, then any solution $x(\varphi)(t)$ of the IVP (3.2.1) and (3.2.2) satisfies

$$\inf_{t \geq 0} x(\varphi)(t) > 0; \tag{3.2.6}$$

(ii) if ($\boldsymbol{H_4}$) is satisfied, then any solution $x(\varphi)(t)$ of the IVP (3.2.1) and (3.2.2) satisfies

$$\sup_{t \geq 0} x(\varphi)(t) < \infty. \tag{3.2.7}$$

Proof. First, we prove part (i). Let $\varphi \in C_+$ be an arbitrary fixed initial function and $x(t) = x(\varphi)(t)$ be any solution of the IVP (3.2.1) and (3.2.2). Then, by Lemma 3.2.2, we have $x(t) > 0$ for $t \geq 0$. Let $T_1 \geq \tau$ and $u_1 > 0$ be defined by ($\mathbf{H_3}$). In virtue of condition ($\mathbf{H_2}$), there exists a positive constant c such that

$$0 < h^{-1}(c) \leq u_1 \qquad \text{and} \qquad \min_{0 \leq t \leq T_1} x(t) > h^{-1}(c) > 0.$$

We show that $x(t) > h^{-1}(c)$ for all $t \geq 0$. Suppose there exists $\bar{t} > T_1$ such that $x(t) > h^{-1}(c)$ for $t \in [0, \bar{t})$ and $x(\bar{t}) = h^{-1}(c)$. Then, using ($\mathbf{H_3}$) with $u = h^{-1}(c)$, we have

$$g(\bar{t}, x_{\bar{t}}) \geq q_1(T_1, h^{-1}(c)) > c,$$

therefore

$$\dot{x}(\bar{t}) = r(\bar{t})\Big(g(\bar{t}, x_{\bar{t}}) - h(x(\bar{t}))\Big) > r(\bar{t})\Big(c - h(h^{-1}(c))\Big) = 0.$$

This is a contradiction, since $\dot{x}(\bar{t}) \leq 0$. Hence $x(t) > h^{-1}(c)$ holds for all $t \geq 0$, so part (i) is proved.

The proof of part (ii) is similar. □

Now we state our main result, which can be used to estimate $\liminf_{t\to\infty} x(t)$ and $\limsup_{t\to\infty} x(t)$. In the next section we will show that in many particular situations these estimations imply that Eq. (3.2.1) is uniformly permanent.

Theorem 3.2.4. *Assume ($\boldsymbol{H_1}$) and ($\boldsymbol{H_2}$) are satisfied. Then for any $\varphi \in C_+$, we have*

(i) if ($\boldsymbol{H_3}$) and ($\boldsymbol{H_5}$) are satisfied, then any solution $x(t) = x(\varphi)(t)$ of the IVP (3.2.1) and (3.2.2) is bounded from below on $[0, \infty)$, and

$$h^{-1}(q_1^*(\underline{x}(\infty))) \leq \underline{x}(\infty); \tag{3.2.8}$$

(ii) if ($\boldsymbol{H_4}$) and ($\boldsymbol{H_6}$) are satisfied, then any solution $x(t) = x(\varphi)(t)$ of the IVP (3.2.1) and (3.2.2) is bounded from above on $[0, \infty)$ and

$$\overline{x}(\infty) \leq h^{-1}(q_2^*(\overline{x}(\infty))). \tag{3.2.9}$$

Proof. First, we prove part (i). Let $x(t)$ be any solution of the IVP (3.2.1) and (3.2.2), and let $T \geq \tau$. By virtue of (3.2.6) we have for any $T \geq \tau$

$$0 < a_{T-\tau} := \inf_{t \geq T-\tau} x(t). \tag{3.2.10}$$

Thus, from (3.2.10) and ($\mathbf{H_5}$), we get

$$g(t, x_t) \geq q_1(T, a_{T-\tau}), \qquad t \geq T.$$

Hence, from (3.2.1), it follows

$$\dot{x}(t) \geq r(t)[q_1(T, a_{T-\tau}) - h(x(t))], \qquad t \geq T. \tag{3.2.11}$$

From (3.2.11) and Theorem 2.1.2 we see that

$$x(t) \geq y(t) \qquad \text{for } t \geq T,$$

where $y(t) = y(T, x(T), q_1(T, a_{T-\tau}))(t)$ is the solution of Eq. (3.2.3) with $c = q_1(T, a_{T-\tau})$ and with the initial condition

$$y(T) = x(T).$$

From Lemma 3.2.1, we see that

$$y(\infty) := \lim_{t\to\infty} y(t) = h^{-1}(q_1(T, a_{T-\tau})).$$

Thus

$$h^{-1}(q_1(T, a_{T-\tau})) = y(\infty) \leq \underline{x}(\infty),$$

and from the last inequality, we have

$$\liminf_{T\to\infty} h^{-1}(q_1(T, a_{T-\tau})) \leq \underline{x}(\infty).$$

But since

$$\underline{x}(\infty) = \lim_{T\to\infty} a_T,$$

then

$$\lim_{T\to\infty} a_{T-\tau} = \underline{x}(\infty). \tag{3.2.12}$$

Using $(\mathbf{H_5})$, (3.2.12) and the strict monotonicity of h^{-1}, we obtain

$$\liminf_{T\to\infty} h^{-1}(q_1(T, a_{T-\tau})) = h^{-1}(\liminf_{T\to\infty} q_1(T, a_{T-\tau})) \geq h^{-1}(q_1^*(\underline{x}(\infty))) \geq 0,$$

and hence

$$h^{-1}(q_1^*(\underline{x}(\infty))) \leq \underline{x}(\infty).$$

Therefore, the proof of (i) is completed.

The proof of part (ii) is similar to the proof of part (i), so it is omitted. $\square$

Our main theorem implies the following corollary, which formulates sufficient conditions for that all positive solutions converge to a constant limit.

Corollary 3.2.5. *Assume all conditions* $(\boldsymbol{H_1})$ – $(\boldsymbol{H_6})$ *hold, moreover* $q^*(w) := q_1^*(w) = q_2^*(w)$ *for* $w \in \mathbb{R}_+$*, and there exists* $u^* > 0$ *such that*

$$q^*(u) > h(u) \quad \text{for } u \in (0, u^*) \qquad \text{and} \qquad q^*(u) < h(u) \quad \text{for } u > u^*. \tag{3.2.13}$$

Then, for any $\varphi \in C_+$*, any solution* $x(t) = x(\varphi)(t)$ *of the IVP (3.2.1) and (3.2.2) satisfies*

$$\lim_{t\to\infty} x(t) = u^*. \tag{3.2.14}$$

Proof. Theorem 3.2.4 yields

$$h^{-1}(q^*(\underline{x}(\infty))) \le \underline{x}(\infty) \le \overline{x}(\infty) \le h^{-1}(q^*(\overline{x}(\infty))),$$

or equivalently,

$$q^*(\underline{x}(\infty)) \le h(\underline{x}(\infty)) \le h(\overline{x}(\infty)) \le q^*(\overline{x}(\infty)).$$

Then condition (3.2.13) implies

$$\overline{x}(\infty) \le u^* \le \underline{x}(\infty),$$

which gives (3.2.14). $\square$

3.3 Applications of the main results

In this section, we provide several corollaries to our main results. First, we consider the equation

$$\dot{x}(t) = \sum_{k=1}^{n} \alpha_k(t) x^p(t - \sigma_k(t)) - \beta(t) x^q(t), \qquad t \ge 0, \tag{3.3.1}$$

with

$$x(t) = \varphi(t), \qquad -\tau \le t \le 0. \tag{3.3.2}$$

A special case ($p = 1$ and $q = 2$) of this equation, a population model with quadratic nonlinearity was studied in [4, 31, 35]. The next result gives explicit estimates for the limit inferior and limit superior of the positive solutions of (3.3.1), which generalize the results of [4, 31].

Corollary 3.3.1. *Consider the IVP (3.3.1) and (3.3.2), where* $0 < p < q$, $q \geq 1$,

$$0 \leq \sigma_k(t) \leq \tau, \qquad t \geq 0 \quad \text{and} \quad k = 1, \dots, n \tag{3.3.3}$$

with some positive constant τ, *and* $\alpha_k, \beta \in C(\mathbb{R}_+, \mathbb{R}_+)$ *with*

$$\beta(t) > 0 \text{ for } t > 0, \quad \int_0^\infty \beta(t)dt = \infty, \quad \lim_{t\to 0+} \frac{\alpha_k(t)}{\beta(t)} < \infty \text{ exists for } k = 1, \dots, n, \tag{3.3.4}$$

and

$$\underline{m} := \liminf_{t\to\infty} \frac{\sum\limits_{k=1}^n \alpha_k(t)}{\beta(t)} > 0 \qquad \text{and} \qquad \overline{m} := \limsup_{t\to\infty} \frac{\sum\limits_{k=1}^n \alpha_k(t)}{\beta(t)} < \infty. \tag{3.3.5}$$

Then, for any initial function $\varphi \in C_+$, *any solution* $x(t) = x(\varphi)(t)$ *of the IVP (3.3.1) and (3.3.2) satisfies*

$$\underline{m}^{\frac{1}{q-p}} \leq \underline{x}(\infty) \leq \overline{x}(\infty) \leq \overline{m}^{\frac{1}{q-p}}. \tag{3.3.6}$$

Proof. The proof is obtained directly from Theorem 3.2.4, where we can rewrite (3.3.1) as follows

$$\dot{x}(t) = \beta(t)\left[\sum_{k=1}^n \frac{\alpha_k(t)}{\beta(t)} x^p(t - \sigma_k(t)) - x^q(t)\right], \qquad t > 0. \tag{3.3.7}$$

Note that (3.3.4) yields that if $\beta(0) = 0$, then the functions $\frac{\alpha_k(t)}{\beta(t)}$ can be extended continuously to $t = 0$. For simplicity, this extended function is denoted by $\frac{\alpha_k(t)}{\beta(t)}$, as well. We can see from (3.3.7) that Eq. (3.3.1) can be written in the form (3.2.1) with $r(t) := \beta(t)$, $g(t, \psi) := \sum\limits_{k=1}^n \frac{\alpha_k(t)}{\beta(t)} \psi^p(-\sigma_k(t))$ and $h(x) := x^q$. Since $q \geq 1$, $h(x)$ is locally Lipschitz continuous, and so conditions $(\mathbf{H_1})$ and $(\mathbf{H_2})$ are satisfied. Now we check that conditions $(\mathbf{H_3})$–$(\mathbf{H_5})$ are satisfied. Suppose that $\psi(s) \geq u > 0$ for $-\tau \leq s \leq 0$, then $g(t, \psi) \geq q_1(T, u)$ for $t \geq T \geq 0$, where

$$q_1(T, u) := m_T u^p, \qquad m_T := \inf_{t \geq T} \frac{\sum\limits_{k=1}^n \alpha_k(t)}{\beta(t)}.$$

Therefore $(\mathbf{H_3})$ is satisfied if $m_{T_1} u^p > u^q$, or equivalently $m_{T_1} > u^{q-p}$ for some

$T_1 \geq \tau$ and small positive u. Since (3.3.5) yields $\underline{m} = \liminf_{T\to\infty} m_T > 0$, there exist $T_1 > 0$ and $u_1 > 0$ such that

$$m_{T_1} > u_1^{q-p} \geq u^{q-p} \quad \text{for } u \in (0, u_1],$$

and hence $(\mathbf{H_3})$ is satisfied. In a similar way we can show that $(\mathbf{C_2})$ is satisfied.

To check $(\mathbf{H_5})$, suppose $v(T) \to w$ as $T \to \infty$. Then

$$\lim_{T\to\infty} q_1(T, v(T)) = \lim_{T\to\infty} m_T v^p(T) = \underline{m} w^p,$$

so $(\mathbf{H_5})$ is satisfied with $q_1^*(w) := \underline{m} w^p$. In a similar way we can check $(\mathbf{H_6})$. Thus Theorem 3.2.4 is applicable, so we see that

$$h^{-1}(q_1^*(\underline{x}(\infty))) \leq \underline{x}(\infty) \leq \overline{x}(\infty) \leq h^{-1}(q_2^*(\overline{x}(\infty))).$$

Hence

$$(\underline{mx}^p(\infty))^{1/q} \leq \underline{x}(\infty) \leq \overline{x}(\infty) \leq (\overline{mx}^p(\infty))^{1/q},$$

therefore we get (3.3.6). □

The next result gives sufficient conditions which yield that all positive solutions are asymptotically equivalent. This result is novel, which is interesting on its own right. One reason for this is that most of the attractivity results in the literature focus on the case when the investigated equation has a saturated equilibrium. See, e.g., [57] Section 4.8 for related results. Corollary 3.3.1 may initiate further research in more general equations without constant steady state solutions.

Corollary 3.3.2. *Consider the IVP (3.3.1) and (3.3.2), where σ_k satisfy (3.3.3), and α_k and β satisfy (3.3.4) and (3.3.5), and suppose $1 \leq p < q$ are integers, and*

$$0 < \frac{\overline{m}}{\underline{m}} < \left(\frac{p}{q}\right)^{\frac{q-1}{q-p}}, \tag{3.3.8}$$

where $\underline{m}$ and $\overline{m}$ are defined in (3.3.5). Then, for any initial functions $\varphi, \psi \in C_+$, any corresponding solutions $x(\varphi)(t)$ and $x(\psi)(t)$ of the IVP (3.3.1) and (3.3.2) satisfy

$$\lim_{t\to\infty} \Big(x(\varphi)(t) - x(\psi)(t) \Big) = 0. \tag{3.3.9}$$

Proof. Introduce the short notations $x_1(t) := x(\varphi)(t)$ and $x_2(t) := x(\psi)(t)$. It follows from Corollary 3.3.1 that

$$\underline{m}^{\frac{1}{q-p}} \le \liminf_{t\to\infty} x_i(t) \le \limsup_{t\to\infty} x_i(t) \le \overline{m}^{\frac{1}{q-p}}, \qquad i=1,2. \tag{3.3.10}$$

Eq. (3.3.1) yields for $t \ge 0$ that

$$\dot{x}_1(t) - \dot{x}_2(t) = \sum_{k=1}^{n} \alpha_k(t)\Big(x_1^p(t-\sigma_k(t)) - x_2^p(t-\sigma_k(t))\Big) - \beta(t)\Big(x_1^q(t) - x_2^q(t)\Big).$$

Therefore the function $w(t) := x_1(t) - x_2(t)$ satisfies

$$\dot{w}(t) = \sum_{k=1}^{n} \alpha_k(t) a_k(t) w(t-\sigma_k(t)) - \beta(t) b(t) w(t), \qquad t \ge 0, \tag{3.3.11}$$

where

$$a_k(t) := \sum_{\ell=0}^{p-1} x_1^{\ell}(t-\sigma_k(t)) x_2^{p-1-\ell}(t-\sigma_k(t)), \quad k=1,\dots,n$$

and

$$b(t) := \sum_{\ell=0}^{q-1} x_1^{\ell}(t) x_2^{q-1-\ell}(t).$$

The definitions of $a_k(t)$, $b(t)$, relation (3.3.10) and assumption (3.3.8) imply

$$\limsup_{t\to\infty} \frac{\sum\limits_{k=1}^{n} \alpha_k(t) a_k(t)}{\beta(t) b(t)} \le \frac{p \cdot \overline{m}^{\frac{p-1}{q-p}}}{q \cdot \underline{m}^{\frac{q-1}{q-p}}} \limsup_{t\to\infty} \frac{\sum\limits_{k=1}^{n} \alpha_k(t)}{\beta(t)} = \frac{p \cdot \overline{m}^{\frac{p-1}{q-p}+1}}{q \cdot \underline{m}^{\frac{q-1}{q-p}}} < 1.$$

Then a simple generalization of Theorem 3.1 of [38] yields that the trivial solution of Eq. (3.3.11) is globally asymptotically stable, so $\lim\limits_{t\to\infty} w(t) = 0$, which completes the proof of the statement. □

Remark 3.3.1. It is interesting to note that if the conditions of Corollary 3.3.2 hold and the IVP (3.3.1) and (3.3.2) has a positive periodic solution, then it is unique and it attracts all positive solutions.

Next, we consider the special case of (3.3.1), which is identical to Eq. (3.1.2)

$$\dot{x}(t) = \sum_{k=1}^{n} \alpha_k(t) x(t-\sigma_k(t)) - \beta(t) x^2(t), \qquad t \ge 0, \tag{3.3.12}$$

with

$$x(t) = \varphi(t), \qquad -\tau \le t \le 0. \tag{3.3.13}$$

Corollary 3.3.1 immediately implies the estimate obtained in [4], but under weaker conditions, since the boundedness conditions (3.1.7) of the coefficients are not required.

Corollary 3.3.3. *Consider the IVP (3.3.12) and (3.3.13), where σ_k satisfy (3.3.3), and α_k and β satisfy (3.3.4) and (3.3.5). Then,*

(i) for any initial function $\varphi \in C_+$, the unique solution $x(t) = x(\varphi)(t)$ of the IVP (3.3.12) and (3.3.13) satisfies

$$\underline{m} \leq \underline{x}(\infty) \leq \overline{x}(\infty) \leq \overline{m}, \tag{3.3.14}$$

where $\underline{m}$ and $\overline{m}$ are defined in (3.3.5).

(ii) Moreover, if in addition

$$\overline{m} < 2\underline{m}, \tag{3.3.15}$$

then any positive solutions of Eq. (3.3.12) are asymptotically equivalent, i.e., (3.3.9) holds.

Next we consider a scalar delay differential equation with more general nonlinearity

$$\dot{x}(t) = \alpha(t) f(x(t - \sigma(t))) - \beta(t) h(x(t)), \qquad t \geq 0, \tag{3.3.16}$$

with

$$x(t) = \varphi(t), \qquad -\tau \leq t \leq 0. \tag{3.3.17}$$

Corollary 3.3.4. *Consider the IVP (3.3.16) and (3.3.17), where the delay function σ satisfies $0 \leq \sigma(t) \leq \tau$ for $t \geq 0$ with some positive constants τ, and $\alpha, \beta \in$*

$C(\mathbb{R}_+, \mathbb{R}_+)$ with

$$\beta(t) > 0 \quad \textit{for } t > 0, \qquad \int_0^\infty \beta(t)dt = \infty, \qquad 0 \leq \lim_{t\to 0+} \frac{\alpha(t)}{\beta(t)} < \infty \textit{ exists}, \tag{3.3.18}$$

and

$$\underline{m} := \liminf_{t\to\infty} \frac{\alpha(t)}{\beta(t)} > 0 \qquad \textit{and} \qquad \overline{m} := \limsup_{t\to\infty} \frac{\alpha(t)}{\beta(t)} < \infty, \tag{3.3.19}$$

$f, h \in C(\mathbb{R}_+, \mathbb{R}_+)$ are increasing functions with $h(0) = 0$, h is locally Lipschitz continuous, and

$$G(u) := \frac{h(u)}{f(u)} \textit{ is monotone increasing,} \quad \lim_{u\to 0} G(u) = 0 \textit{ and } \lim_{u\to\infty} G(u) = \infty. \tag{3.3.20}$$

Then, for any initial function $\varphi \in C_+$, any solution $x(t) = x(\varphi)(t)$ of the IVP (3.3.16) and (3.3.17) satisfies

$$G^{-1}(\underline{m}) \leq \underline{x}(\infty) \leq \overline{x}(\infty) \leq G^{-1}(\overline{m}). \tag{3.3.21}$$

Proof. We rewrite (3.3.16) as

$$\dot{x}(t) = \beta(t)\left[\frac{\alpha(t)}{\beta(t)} f(x(t-\sigma(t))) - h(x(t))\right], \qquad t \geq 0. \tag{3.3.22}$$

We can see from (3.3.22) that $r(t) := \beta(t)$ and $g(t,\psi) := \frac{\alpha(t)}{\beta(t)} f(\psi(-\sigma(t)))$. It is clear that conditions ($\mathbf{H_1}$) and ($\mathbf{H_2}$) hold. We check that conditions ($\mathbf{H_3}$)–($\mathbf{H_6}$) are satisfied. Suppose that $\psi(s) \geq u > 0$ for $-\tau \leq s \leq 0$, then $g(t, x_t) \geq q_1(T, u)$ for $t \geq T$, where

$$q_1(T, u) := m_T f(u), \qquad m_T := \inf_{t\geq T} \frac{\alpha(t)}{\beta(t)}.$$

Hence ($\mathbf{H_3}$) is satisfied if $m_{T_1} f(u) > h(u)$, or equivalently

$$m_{T_1} > G(u) \tag{3.3.23}$$

for some $T_1 \geq \tau$ and for small enough positive u. It follows from (3.3.19) that there exists $T_1 > 0$ such that $m_{T_1} > 0$. Using $\lim_{u\to 0} G(u) = 0$, there exists $u_1 > 0$ such that $0 < G(u_1) < m_{T_1}$. Thus we have that (3.3.23) holds for $u \in (0, u_1]$, and hence ($\mathbf{H_3}$) is satisfied. Similarly, we can check ($\mathbf{H_4}$).

To show ($\mathbf{H_5}$), suppose that $\lim_{T\to\infty} v(T) = w$, and consider

$$\lim_{T\to\infty} q_1(T, v(T)) = \lim_{T\to\infty} m_T f(v(T)) = \underline{m} f(w),$$

so ($\mathbf{H_5}$) is satisfied with $q_1^*(w) := \underline{m} f(w)$. In a similar way we can check ($\mathbf{H_6}$). Thus Theorem 3.2.4 is applicable, so we see that

$$h^{-1}(q_1^*(\underline{x}(\infty))) \le \underline{x}(\infty) \le \overline{x}(\infty) \le h^{-1}(q_2^*(\overline{x}(\infty))).$$

Hence

$$\underline{m} f(\underline{x}(\infty)) \le h(\underline{x}(\infty)) \le h(\overline{x}(\infty)) \le \overline{m} f(\overline{x}(\infty)),$$

and therefore, using (3.3.20), we get (3.3.21). □

Corollary 3.3.5. *Suppose all conditions of Corollary 3.3.4 hold, moreover*

$$0 < m := \lim_{t\to\infty} \frac{\alpha(t)}{\beta(t)} < \infty \tag{3.3.24}$$

exists, and there exists $u^* > 0$ such that

$$mf(u) > h(u) \quad \text{for } u \in (0, u^*) \qquad \text{and} \qquad mf(u) < h(u) \quad \text{for } u > u^*. \tag{3.3.25}$$

Then, for any initial function $\varphi \in C_+$, any solution $x(t) = x(\varphi)(t)$ of the IVP (3.3.16) and (3.3.17) satisfies

$$\lim_{t\to\infty} x(t) = u^*. \tag{3.3.26}$$

Proof. It follows from the proof of Corollary 3.3.4 that $q_1^*(w) = q_2^*(w) = mf(w)$, $w \in \mathbb{R}_+$, so Corollary 3.2.5 yields (3.3.26). □

Now we consider the IVP

$$\dot{x}(t) = \alpha(t) \int_{-\tau}^{0} f(s, x(t+s))\, ds - \beta(t) h(x(t)), \qquad t \ge 0 \tag{3.3.27}$$

with the initial condition

$$x(t) = \varphi(t), \qquad -\tau \le t \le 0. \tag{3.3.28}$$

Corollary 3.3.6. *Consider the IVP (3.3.27) and (3.3.28), where $\alpha, \beta \in C(\mathbb{R}_+, \mathbb{R}_+)$ obey (3.3.18) and (3.3.19), $f \in C([-\tau, 0] \times \mathbb{R}, \mathbb{R}_+)$ is increasing in its second vari-*

able, $h \in C(\mathbb{R}_+, \mathbb{R}_+)$ is an increasing function with $h(0) = 0$, h is locally Lipschitz continuous, and

$$G(u) := \frac{h(u)}{\int_{-\tau}^{0} f(s,u)\,ds} \text{ is monotone increasing, } \quad \lim_{u\to 0} G(u) = 0, \ \lim_{u\to\infty} G(u) = \infty.$$

Then, for any initial function $\varphi \in C_+$, any solution $x(t) = x(\varphi)(t)$ of the IVP (3.3.27) and (3.3.28) satisfies

$$G^{-1}(\underline{m}) \le \underline{x}(\infty) \le \overline{x}(\infty) \le G^{-1}(\overline{m}). \tag{3.3.29}$$

Proof. The proof is similar to that of Corollary 3.3.4, so it is omitted. □

Next we consider the IVP

$$\dot{x}(t) = \frac{\alpha(t)x(t-\sigma(t))}{1+\gamma(t)x(t-\sigma(t))} - \beta(t)x^2(t), \qquad t \ge 0, \tag{3.3.30}$$

with the initial condition

$$x(t) = \varphi(t), \qquad -\tau \le t \le 0. \tag{3.3.31}$$

This is a special case of the alternative delayed logistic population model introduced in [2] (see also [4, 31]).

We show that, under weak conditions on the coefficients, Theorem 3.2.4 is applicable to estimate $\underline{x}(\infty)$ and $\overline{x}(\infty)$.

Corollary 3.3.7. *Suppose $0 \le \sigma(t) \le \tau$ with some $\tau > 0$, the coefficients $\alpha, \beta, \gamma \in C(\mathbb{R}_+, \mathbb{R}_+)$ with*

$$\beta(t) > 0, \ t > 0, \quad \int_0^\infty \beta(t)dt = \infty, \quad \lim_{t\to 0+} \frac{\alpha(t)}{\beta(t)} < \infty \text{ exists}, \quad 0 < \liminf_{t\to\infty} \gamma(t), \tag{3.3.32}$$

and for some $\varepsilon > 0$

$$\underline{m}_\varepsilon := \liminf_{t\to\infty} \frac{-1+\sqrt{1+\frac{4\alpha(t)\gamma(t)}{(1+\varepsilon)\beta(t)}}}{2\gamma(t)} > 0 \tag{3.3.33}$$

and

$$\overline{m}_\varepsilon := \limsup_{t\to\infty} \frac{-1+\sqrt{1+\frac{4\alpha(t)\gamma(t)}{(1+\varepsilon)\beta(t)}}}{2\gamma(t)} < \infty. \tag{3.3.34}$$

Furthermore, suppose there exist functions q_1^* and q_2^* so that if $\lim_{T\to\infty} v(T) = w$, then

$$\liminf_{T\to\infty} \inf_{t\geq T} \frac{\frac{\alpha(t)}{\beta(t)} v(T)}{1+\gamma(t)v(T)} \geq q_1^*(w) \tag{3.3.35}$$

and

$$\limsup_{T\to\infty} \sup_{t\geq T} \frac{\frac{\alpha(t)}{\beta(t)} v(T)}{1+\gamma(t)v(T)} \leq q_2^*(w). \tag{3.3.36}$$

Then, for any initial function $\varphi \in C_+$, the solution $x(t) = x(\varphi)(t)$ of the IVP (3.3.30) and (3.3.31) satisfies

$$\sqrt{q_1^*(\underline{x}(\infty))} \leq \underline{x}(\infty) \leq \overline{x}(\infty) \leq \sqrt{q_2^*(\overline{x}(\infty))}. \tag{3.3.37}$$

Proof. We can rewrite (3.3.30) as follows

$$\dot{x}(t) = \beta(t)\left[\frac{\frac{\alpha(t)}{\beta(t)} x(t-\sigma(t))}{1+\gamma(t)x(t-\sigma(t))} - x^2(t)\right], \qquad t \geq 0,$$

where $\frac{\alpha(t)}{\beta(t)}$ denotes the continuous extension of the function to $t = 0$ if $\beta(0) = 0$. Let us define $r(t) := \beta(t)$, $g(t,\psi) := \frac{\frac{\alpha(t)}{\beta(t)}\psi(-\sigma(t))}{1+\gamma(t)\psi(-\sigma(t))}$ and $h(x) := x^2$. It is clear that conditions $(\mathbf{H_1})$ and $(\mathbf{H_2})$ are satisfied. We check that conditions $(\mathbf{H_3})$–$(\mathbf{H_4})$ are satisfied. Suppose that $\psi(s) \geq u > 0$ for $-\tau \leq s \leq 0$, then $g(t, x_t) \geq q_1(T, u)$ for $t \geq T$, where

$$q_1(T,u) := \inf_{t\geq T} \frac{\frac{\alpha(t)}{\beta(t)} u}{1+\gamma(t)u}.$$

Thus $(\mathbf{H_3})$ is satisfied if for some $T_1 \geq \tau$, $\varepsilon > 0$ and small enough $u > 0$

$$\frac{\frac{\alpha(t)}{\beta(t)} u}{1+\gamma(t)u} \geq (1+\varepsilon)u^2, \qquad t \geq T_1,$$

or equivalently

$$(1+\varepsilon)\gamma(t)u^2 + (1+\varepsilon)u - \frac{\alpha(t)}{\beta(t)} \leq 0, \qquad t \geq T_1. \tag{3.3.38}$$

Relation (3.3.32) implies there exists $T_1 \geq \tau$ and $\varepsilon > 0$ such that $\gamma(t) > 0$ for $t \geq T_1$ and

$$u_1 := \inf_{t\geq T_1} \frac{-1+\sqrt{1+\frac{4\alpha(t)\gamma(t)}{(1+\varepsilon)\beta(t)}}}{2\gamma(t)} > 0. \tag{3.3.39}$$

So for $t \geq T_1$

$$(1+\varepsilon)\gamma(t)y^2 + (1+\varepsilon)y - \frac{\alpha(t)}{\beta(t)} = 0$$

is a quadratic equation, and (3.3.33) yields that it has a negative solution and a positive solution

$$\frac{-1+\sqrt{1+\frac{4\alpha(t)\gamma(t)}{(1+\varepsilon)\beta(t)}}}{2\gamma(t)}.$$

Therefore (3.3.39) yields (3.3.38), and hence $q_1(T_1, u) > u^2$ holds for $0 < u \leq u_1$. In a similar way we can show that $(\mathbf{H_4})$ is satisfied.

Assumption $(\mathbf{H_5})$ follows from (3.3.35), since

$$\liminf_{T\to\infty} q_1(T, v(T)) = \liminf_{T\to\infty} \inf_{t\geq T} \frac{\frac{\alpha(t)}{\beta(t)}v(T)}{1+\gamma(t)v(T)}.$$

Assumption $(\mathbf{H_6})$ can be shown similarly. Then Theorem 3.2.4 yields (3.3.37). $\square$

The next two corollaries illustrate two cases when relations (3.3.35) and (3.3.36) can be checked easily. First consider the case when $\gamma(t) \to \infty$ as $t \to \infty$.

Corollary 3.3.8. *Suppose $0 \leq \sigma(t) \leq \tau$ with some $\tau > 0$, the coefficients $\alpha, \beta, \gamma \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfy (3.3.32), (3.3.33) and (3.3.34). Furthermore, suppose*

$$\lim_{t\to\infty} \gamma(t) = \infty. \tag{3.3.40}$$

Then, for any initial function $\varphi \in C_+$, the solution $x(t) = x(\varphi)(t)$ of the IVP (3.3.30) and (3.3.31) satisfies

$$\underline{m} \leq \underline{x}(\infty) \leq \overline{x}(\infty) \leq \overline{m}, \tag{3.3.41}$$

where $\underline{m} := \underline{m}_0$ and $\overline{m} := \overline{m}_0$ are defined in (3.3.33) and (3.3.34) with $\varepsilon = 0$.

Proof. Assumption (3.3.40) yields

$$\begin{aligned}
\underline{m}_\varepsilon &= \liminf_{t\to\infty} \frac{-1+\sqrt{1+\frac{4\alpha(t)\gamma(t)}{(1+\varepsilon)\beta(t)}}}{2\gamma(t)} \\
&= \liminf_{t\to\infty} \Big(-\frac{1}{2\gamma(t)} + \sqrt{\frac{1}{4\gamma^2(t)} + \frac{\alpha(t)}{(1+\varepsilon)\beta(t)\gamma(t)}}\Big) \\
&= \liminf_{t\to\infty} \sqrt{\frac{\alpha(t)}{(1+\varepsilon)\beta(t)\gamma(t)}} \\
&= \frac{1}{\sqrt{1+\varepsilon}}\underline{m},
\end{aligned}$$

and similarly,

$$\overline{m}_\varepsilon = \limsup_{t\to\infty} \frac{-1+\sqrt{1+\frac{4\alpha(t)\gamma(t)}{(1+\varepsilon)\beta(t)}}}{2\gamma(t)} = \limsup_{t\to\infty} \sqrt{\frac{\alpha(t)}{(1+\varepsilon)\beta(t)\gamma(t)}} = \frac{1}{\sqrt{1+\varepsilon}}\overline{m}.$$

To check (3.3.35), suppose $\lim\limits_{T\to\infty} v(T) = w$, and let $\varepsilon > 0$ be fixed. Then, for large enough t, we have $\frac{\alpha(t)}{\beta(t)\gamma(t)} \ge \underline{m}_\varepsilon^2$. Then we have

$$\begin{aligned}
\liminf_{T\to\infty} \inf_{t\ge T} \frac{\frac{\alpha(t)}{\beta(t)}v(T)}{1+\gamma(t)v(T)} &= \liminf_{T\to\infty} \inf_{t\ge T} \frac{\frac{\alpha(t)}{\beta(t)\gamma(t)}v(T)}{\frac{1}{\gamma(t)}+v(T)} \\
&\ge \liminf_{T\to\infty} \frac{\underline{m}_\varepsilon^2 v(T)}{\frac{1}{\inf_{t\ge T}\gamma(t)}+v(T)} \\
&= \underline{m}_\varepsilon^2.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, it follows

$$\liminf_{T\to\infty} \inf_{t\ge T} \frac{\frac{\alpha(t)}{\beta(t)}v(T)}{1+\gamma(t)v(T)} \ge \underline{m}^2,$$

i.e., $q_1^*(w) = \underline{m}^2$ can be selected in $(\mathbf{D_1})$. Similar calculation shows that

$$\limsup_{T\to\infty} \sup_{t\ge T} \frac{\frac{\alpha(t)}{\beta(t)}v(T)}{1+\gamma(t)v(T)} \le \overline{m}^2.$$

Then Theorem 3.2.4 yields (3.3.41). $\square$

In the case when $\gamma(t)$ and $\frac{\alpha(t)}{\beta(t)}$ are bounded, we can give an explicit estimates in (3.3.35) and (3.3.36), so we obtain explicit estimates of $\underline{x}(\infty)$ and $\overline{x}(\infty)$.

Corollary 3.3.9. *Suppose $0 \le \sigma(t) \le \tau$ with some $\tau > 0$, and the coefficients $\alpha, \beta, \gamma \in C([0,\infty), \mathbb{R}_+)$ satisfy (3.3.32). Moreover, suppose*

$$0 < \underline{m} := \liminf_{t\to\infty} \frac{\alpha(t)}{\beta(t)} \le \limsup_{t\to\infty} \frac{\alpha(t)}{\beta(t)} =: \overline{m} < \infty,$$

and

$$0 < \underline{l} := \liminf_{t\to\infty} \gamma(t) \le \limsup_{t\to\infty} \gamma(t) =: \bar{l} < \infty.$$

Then the solutions of the IVP (3.3.30) and (3.3.31) with $\varphi \in C_+$ *satisfy*

$$\frac{-1+\sqrt{1+4\underline{m}\bar{l}}}{2\bar{l}} \le \underline{x}(\infty) \le \overline{x}(\infty) \le \frac{-1+\sqrt{1+4\overline{m}\underline{l}}}{2\underline{l}}. \tag{3.3.42}$$

Proof. To check (3.3.35) we consider

$$\lim_{T\to\infty} \inf_{t\ge T} \frac{\frac{\alpha(t)}{\beta(t)} v(T)}{1+\gamma(t)v(T)} \ge \lim_{T\to\infty} \frac{\inf_{t\ge T} \frac{\alpha(t)}{\beta(t)} v(T)}{1+\sup_{t\ge T}\gamma(t)v(T)} = \frac{\underline{m}v(T)}{1+\bar{l}v(T)},$$

so (3.3.35) holds with

$$q_1^*(w) = \frac{\underline{m}w}{1+\bar{l}w}.$$

Similarly, the function

$$q_2^*(w) = \frac{\overline{m}w}{1+\underline{l}w}$$

satisfies (3.3.36). Then (3.3.37) implies (3.3.42). □

Finally we consider

$$\dot{x}(t) = \sum_{k=1}^{n} \frac{\alpha_k(t)x(t-\sigma_k(t))}{1+\gamma_k(t)x(t-\sigma_k(t))} - a\beta(t)x(t) - \beta(t)x^2(t), \qquad t \ge 0, \tag{3.3.43}$$

where $a > 0$, and we associate the initial condition

$$x(t) = \varphi(t), \qquad -\tau \le t \le 0. \tag{3.3.44}$$

Note that a slightly more general version of Eq (3.3.43) was studied in [31] where $a\beta(t)$ was replaced by a function $\mu(t)$.

Corollary 3.3.10. *Suppose* $a > 0$, $0 \le \sigma_k(t) \le \tau$ *with some* $\tau > 0$, *and the coefficients* $\alpha_k, \beta, \gamma_k \in C([0,\infty), \mathbb{R}_+)$ $(k = 1, \dots, n)$ *satisfy (3.3.32). Moreover, suppose*

$$0 < \underline{m}_k := \liminf_{t\to\infty} \frac{\alpha_k(t)}{\beta(t)} \le \limsup_{t\to\infty} \frac{\alpha_k(t)}{\beta(t)} =: \overline{m}_k < \infty,$$

$$0 < \underline{l} := \min_{k=1,\dots,n} \liminf_{t\to\infty} \gamma_k(t) \le \max_{k=1,\dots,n} \limsup_{t\to\infty} \gamma_k(t) =: \bar{l} < \infty$$

and

$$\sum_{k=1}^{n} \underline{m}_k > a.$$

Then the solutions of the IVP (3.3.43) and (3.3.44) with $\varphi \in C_+$ *satisfy*

$$\frac{-(1+a\bar{l}) + \sqrt{(1+a\bar{l})^2 - 4\bar{l}(a - \sum_{k=1}^{n} \underline{m}_k)}}{2\bar{l}} \leq \underline{x}(\infty) \tag{3.3.45}$$

and

$$\overline{x}(\infty) \leq \frac{-(1+a\underline{l}) + \sqrt{(1+a\underline{l})^2 - 4\underline{l}(a - \sum_{k=1}^{n} \overline{m}_k)}}{2\underline{l}}. \tag{3.3.46}$$

Proof. The proof is similar to that of Corollary 3.3.9 using the function $h(u) = au + u^2$, so it is omitted here. □

3.4 Examples

In this section, we provide several examples to our main results.

Example 3.4.1. Consider the differential equation

$$\dot{x}(t) = t(2+\cos t)x(t-2.5) - tx^2(t), \qquad t \geq 0. \tag{3.4.1}$$

It is clear that (3.4.1) is a special case of (3.3.12) with $n = 1$, $\alpha_1(t) = t(2+\cos t)$, $\beta(t) = t$, and relations (3.3.4) and (3.3.5) hold. We get

$$\underline{m} = \liminf_{t\to\infty} \frac{\alpha_1(t)}{\beta(t)} = \liminf_{t\to\infty}(2+\cos t) = 1,$$
$$\overline{m} = \limsup_{t\to\infty} \frac{\alpha_1(t)}{\beta(t)} = \limsup_{t\to\infty}(2+\cos t) = 3.$$

Hence Corollary 3.3.3 yields that all solutions of (3.4.1) corresponding to an initial function $\varphi \in C_+$ satisfy

$$1 \leq \underline{x}(\varphi)(\infty) \leq \overline{x}(\varphi)(\infty) \leq 3.$$

We note that the results of [4] and [31] cannot be applied for (3.4.1), since the coefficients do not satisfy (3.1.7).

In Figure 3.4.1 we plotted the solutions of Eq. 3.4.1 starting from the constant initial functions $\varphi(t) = 0.2$, $\varphi(t) = 1$ and $\varphi(t) = 2$. We can see from the figure (and from other numerical runnings) that the above estimates hold, moreover, all solutions seem to be asymptotically equivalent, despite of that condition (3.3.15) does not hold in this example. $\square$

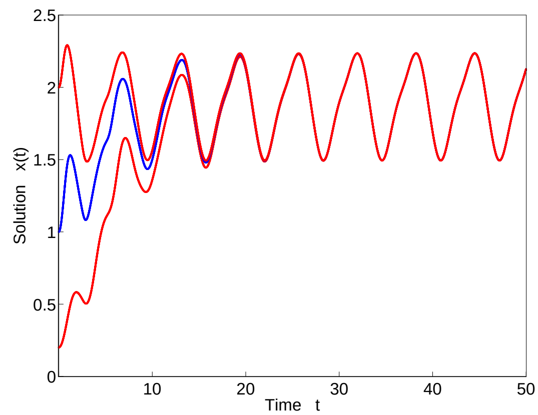

Figure 3.4.1: Solutions of Eq. (3.4.1) corresponding to the initial functions $\varphi(t) = 0.2$, $\varphi(t) = 1$ and $\varphi(t) = 2$

The next example shows that estimate (3.3.14) is sharp in some cases.

Example 3.4.2. For $\tau > \pi$ consider the differential equation

$$\dot{x}(t) = \left(\frac{\pi}{\tau}\sin\frac{4\pi}{\tau}t + e^{\frac{1}{2}\sin^2\frac{2\pi}{\tau}t}\right)x(t-\tau) - x^2(t), \qquad t \geq 0. \tag{3.4.2}$$

An application of Corollary 3.3.3 gives that the solutions of (3.4.2) corresponding to an initial function $\varphi \in C_+$ satisfy

$$\underline{m}_\tau \leq \underline{x}(\infty) \leq \overline{x}(\infty) \leq \overline{m}_\tau,$$

where

$$\underline{m}_\tau := \liminf_{t\to\infty}\left(\frac{\pi}{\tau}\sin\frac{4\pi}{\tau}t + e^{\frac{1}{2}\sin^2\frac{2\pi}{\tau}t}\right)$$

and

$$\overline{m}_\tau := \limsup_{t\to\infty} \left(\frac{\pi}{\tau} \sin \frac{4\pi}{\tau} t + e^{\frac{1}{2}\sin^2 \frac{2\pi}{\tau} t} \right).$$

Simple calculation shows that the function

$$x(t) = e^{\frac{1}{2}\sin^2 \frac{2\pi}{\tau} t}, \qquad t \geq 0$$

is a positive solution of Eq. (3.4.2), and $\underline{x}(\infty) = 1$, $\overline{x}(\infty) = \sqrt{e}$. Therefore for $\tau > \pi$

$$\underline{m}_\tau \leq 1 < \sqrt{e} \leq \overline{m}_\tau.$$

It is easy to see that $\underline{m}_\tau \to 1$ and $\overline{m}_\tau \to \sqrt{e}$ as $\tau \to \infty$, so our estimations are getting sharper and sharper as $\tau \to \infty$, see Figure 3.4.2.

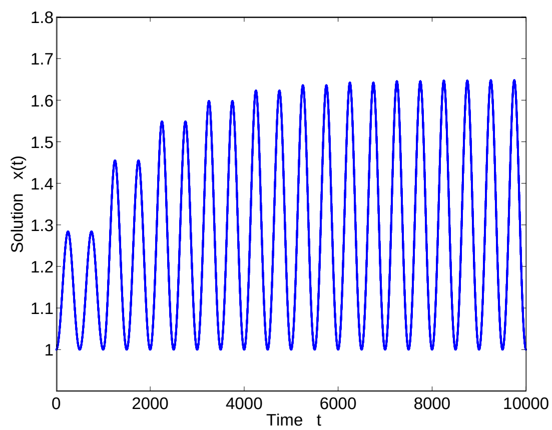

Figure 3.4.2: Solution of Eq. (3.4.2) corresponding to the initial function $\varphi(t) = 1$ and $\tau = 1000$

We note that condition (3.3.15) holds for large enough τ, so then Remark 3.3.1 yields immediately that for such τ the function $e^{\frac{1}{2}\sin^2 \frac{2\pi}{\tau} t}$ is the only positive periodic solution of (3.4.2), and it attracts all positive solutions. $\square$

Example 3.4.3. Consider

$$\dot{x}(t) = t\Big((2 + \frac{1}{t+1})x(t - 3 - \sin t) - x^2(t)\Big), \qquad t \geq 0. \tag{3.4.3}$$

All conditions of Corollary 3.3.5 hold with $m = 2$ and $u^* = 2$, so the solutions of (3.4.3), as shown in Figure 3.4.3, corresponding to an initial function $\varphi \in C_+$

satisfies

$$\lim_{t\to\infty} x(t) = 2.$$

$\square$

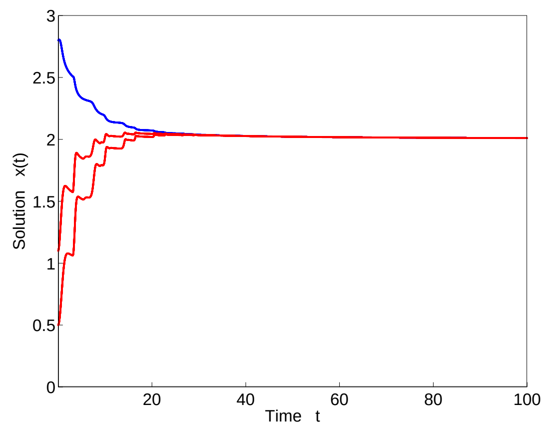

Figure 3.4.3: Solution of Eq. (3.4.3) corresponding to the initial functions $\varphi(t) = 0.5, \varphi(t) = 1.1$ and $\varphi(t) = 2.8$

Example 3.4.4. Consider the equation

$$\dot{x}(t) = \frac{(1+\cos^2 t)x(t-3)}{1+t(\delta+\sin^2 t)x(t-3)} - \frac{1}{t+1}x^2(t), \qquad t \ge 0, \tag{3.4.4}$$

where $\delta > 0$ with the initial condition (3.3.31), i.e., let $\alpha(t) = 1+\cos^2 t$, $\beta(t) = \frac{1}{t+1}$ and $\gamma(t) = t(\delta+\sin^2 t)$ in (3.3.30). Clearly, relation (3.3.32) holds. To check (3.3.33) with $\varepsilon = 0$, we have

$$\begin{aligned}
\liminf_{t\to\infty} \frac{-1+\sqrt{1+\frac{4\alpha(t)\gamma(t)}{\beta(t)}}}{2\gamma(t)} &= \liminf_{t\to\infty}\Big(-\frac{1}{2\gamma(t)} + \sqrt{\frac{1}{4\gamma^2(t)} + \frac{\alpha(t)}{\beta(t)\gamma(t)}}\Big) \\
&= \liminf_{t\to\infty} \sqrt{\frac{\alpha(t)}{\beta(t)\gamma(t)}} \\
&= \liminf_{t\to\infty} \sqrt{\frac{(1+\cos^2 t)(t+1)}{t(\delta+\sin^2 t)}} \\
&\ge \sqrt{\frac{1}{\delta+1}}.
\end{aligned}$$

Similarly, (3.3.34) holds since

$$\limsup_{t\to\infty} \frac{-1+\sqrt{1+\frac{4\alpha(t)\gamma(t)}{\beta(t)}}}{2\gamma(t)} = \limsup_{t\to\infty} \sqrt{\frac{(1+\cos^2 t)(t+1)}{t(\delta+\sin^2 t)}} \le \sqrt{\frac{2}{\delta}}.$$

Then Corollary 3.3.8 yields the solutions corresponding to initial function $\varphi \in C_+$ satisfy

$$\sqrt{\frac{1}{\delta+1}} \le \underline{x}(\infty) \le \overline{x}(\infty) \le \sqrt{\frac{2}{\delta}}.$$

For $\delta = 0.8$ the above estimates give $0.7454 \le \underline{x}(\infty) \le \overline{x}(\infty) \le 1.5811$. In Figure 3.4.4 we display numerically generated solutions using the initial functions $\varphi(t) = 0.2$, $\varphi(t) = 0.5$ and $\varphi(t) = 2$. These runnings indicate that the solutions are asymptotically equivalent.

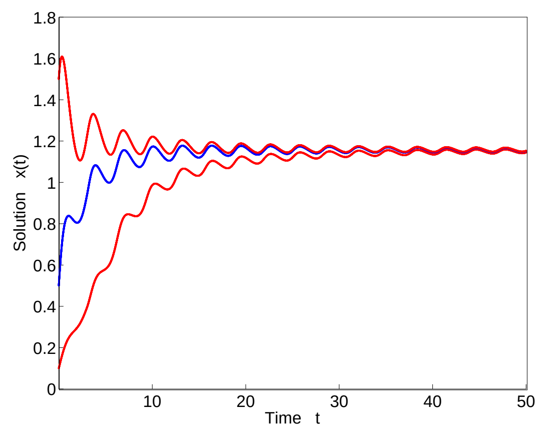

Figure 3.4.4: Solutions of Eq. (3.4.4) corresponding to $\delta = 0.8$ and the initial functions $\varphi(t) = 0.1$, $\varphi(t) = 0.5$ and $\varphi(t) = 1.5$.

□

Example 3.4.5. Consider the differential equation

$$\dot{x}(t) = \frac{t(3+\cos t + \frac{4}{2t+1})x(t-2)}{1+(2+\sin t)x(t-2)} - tx^2(t), \qquad t \ge 0 \tag{3.4.5}$$

with the initial condition (3.3.31). Then we see that

$$\begin{aligned}
\underline{m} &:= \liminf_{t\to\infty} \frac{\alpha(t)}{\beta(t)} = \liminf_{t\to\infty} \left[3 + \cos t + \frac{4}{2t+1}\right] = 2,\\
\overline{m} &:= \limsup_{t\to\infty} \frac{\alpha(t)}{\beta(t)} = \limsup_{t\to\infty} \left[3 + \cos t + \frac{4}{2t+1}\right] = 4,\\
\underline{l} &:= \liminf_{t\to\infty} \gamma(t) = \liminf_{t\to\infty} (2 + \sin t) = 1
\end{aligned}$$

and

$$\bar{l} := \limsup_{t\to\infty} \gamma(t) = \limsup_{t\to\infty} (2 + \sin t) = 3.$$

Substituting in (3.3.42) we find that

$$0.66666\ldots = \frac{-1+\sqrt{25}}{6} \le \underline{x}(\infty) \le \overline{x}(\infty) \le \frac{-1+\sqrt{17}}{2} = 1.56155\ldots.$$

As it is shown in Figure 3.4.5. $\square$

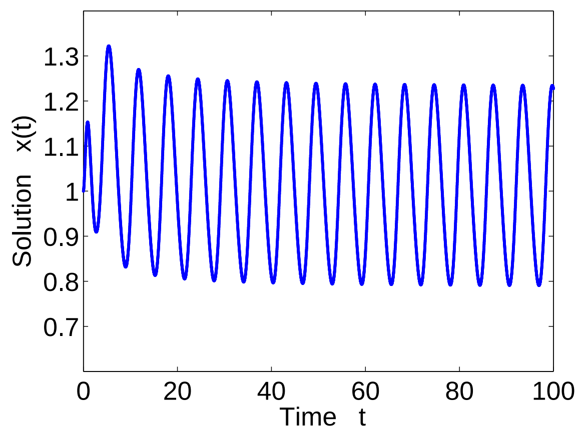

Figure 3.4.5: Solution of Eq. (3.4.5) corresponding to the initial functions $\varphi(t) = 1$

Example 3.4.6. Consider the differential equation

$$\dot{x}(t) = \frac{(2+\sin t)x(t-\tau)}{1+x(t-\tau)} - ax(t) - x^2(t), \qquad t \ge 0 \tag{3.4.6}$$

with the initial condition (3.3.44). Here $n = 1$, $\alpha_1 = 2 + \sin t$, $\gamma_1(t) = 1$, $\beta(t) = 1$, and so $\underline{l} = \bar{l} = 1$, $\underline{m}_1 = 1$ and $\overline{m}_1 = 3$.

Consider first the case when $a = 0.1$. Then (3.3.45) and (3.3.46) yield the estimates

$$0.5 \leq \underline{x}(\infty) \leq \overline{x}(\infty) \leq 1.2114.$$

Note that Theorem 3.2 of [31] yields the estimates

$$0.45 = \frac{\sum\limits_{k=1}^{n} \inf\limits_{t\geq 0} \alpha_k(t) - a \sup\limits_{t\geq 0} \beta(t)}{\sup\limits_{t\geq 0} \beta(t) + \sum\limits_{k=1}^{n} \inf\limits_{t\geq 0} \alpha_k(t) \sup\limits_{t\geq 0} \gamma_k(t)} \leq \underline{x}(\infty)$$

and

$$\overline{x}(\infty) \leq \limsup_{t\to\infty} \frac{1}{\beta(t)} \sum_{k=1}^{n} \alpha_k(t) - a = 2.9,$$

so for this example our result gives better estimates.

Next consider the case when $a = 0.2$. Then our estimates (3.3.45) and (3.3.46) give

$$0.3798 \leq \underline{x}(\infty) \leq \overline{x}(\infty) \leq 1.1204.$$

If we apply Theorem 3.2 of [31] then we get the estimates

$$0.4 \leq \underline{x}(\infty) \leq \overline{x}(\infty) \leq 2.8,$$

where the lower estimate is better than ours, but the upper estimate is worse. $\square$

Chapter 4

Existence and uniqueness of positive solutions of a system of nonlinear algebraic equations

In this chapter we consider the nonlinear system $\gamma_i(x_i) = \sum_{j=1}^{n} g_{ij}(x_j), \ \ 1 \le i \le n$. We give sufficient conditions which imply the existence and uniqueness of positive solutions of the system. Our theorem extends earlier results known in the literature. Also, we give several examples to illustrate the main result of this chapter. This existence and uniqueness condition will be essential in the proof of our results in Chapter 5.

4.1 Introduction

Nonlinear or linear algebraic systems appear as steady-state equations in continuous and discrete dynamical models (e.g., reaction-diffusion equations [51, 58], neural networks [17, 18, 53, 67] compartmental systems [11, 15, 39, 41, 54, 55], population models [49, 63]). Next we mention some typical models.

Compartmental systems are used to model many processes in pharmacokinetics, metabolism, epidemiology and ecology. We refer to [54, 55] as surveys of basic theory and applications of linear and nonlinear compartmental system without and with delays. A standard form of a linear compartmental system with delays is

$$\dot{q}_i(t) = -k_{ii}q_i(t) + \sum_{\substack{j=1\\ j\neq i}}^{n} k_{ij}q_j(t-\tau_{ij}) + I_i, \qquad i = 1,\ldots,n. \tag{4.1.1}$$

Here $q_i(t)$ is the mass of the ith compartment at time t, $k_{ij} > 0$ represent the transfer or rate coefficients, $I_i \geq 0$ is the inflow to the ith compartment. A possible generalization of (4.1.1) used in several applications is a compartmental system, where it is assumed that the intercompartmental flows are functions of the state of the donor compartments only in the form $k_{ij}f_j(q_j)$ with some positive nonlinear function f_j. So we get the nonlinear donor-controlled compartmental system (see, e.g., [11, 14])

$$\dot{q}_i(t) = -k_{ii}f_i(q_i(t)) + \sum_{\substack{j=1\\ j\neq i}}^{n} k_{ij}f_j(q_j(t-\tau_{ij})) + I_i, \qquad i = 1,\ldots,n. \tag{4.1.2}$$

Next we consider an ecological system of n species which are living in a symbiotic relationship with the other species (see [34]):

$$\dot{x}_i = x_i\left(-k_{ii}x_i + \sum_{\substack{j=1\\ j\neq i}}^{n} k_{ij}x_j + b_i\right), \qquad i = 1,\ldots,n. \tag{4.1.3}$$

Here $k_{ii} > 0$ represents the measure of the mortality due to intraspecific competition, the terms $b_i \geq 0$ represents the per capita growth due to external (inexhaustible) sources of energy, and the coefficients k_{ij} $(j \neq i)$ are nonnegative due to the symbiosis.

Cellular neural networks were introduced by Chua and Yang [19] in 1988, and since then they have been applied in many scientific and engineering applications. Here we consider the Hopfield neural network studied in [17]

$$C_i\dot{u}_i = \sum_{j=1}^{n} T_{ij}g_j(u_j) - \frac{u_i}{R_i} + I_i, \qquad i = 1,\ldots,n, \tag{4.1.4}$$

where $C_i > 0$, $R_i > 0$ and I_i are capacity, resistance, bias, respectively, T_{ij} is the interconnection weight, and g_i is a strictly monotone increasing nonlinear function with $g_i(0) = 0$.

Finally, we recall the delayed Cohen–Grossberg neural network model from [53]

$$\dot{x}_i(t) = -d_i(x_i(t))\left(c_i(x_i(t)) - \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) - \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_{ij}(t))) + J_i\right) \tag{4.1.5}$$

for $i = 1, \ldots, n$.

A nonzero equilibrium of both (4.1.1) and (4.1.3) satisfies a linear system of the form

$$A\mathbf{x} = \mathbf{b}, \tag{4.1.6}$$

where $A \in \mathbb{R}^{n\times n}$ has elements

$$a_{ij} = \begin{cases} k_{ii}, & j = i, \\ -k_{ij}, & j \neq i, \end{cases}$$

and $\mathbf{b} \geq \mathbf{0}$, i.e., all coordinates of $\mathbf{b}$ are nonnegative. It is known (see, e.g., [10]) that if A is a nonsingular M-matrix and $\mathbf{b} \gg \mathbf{0}$, i.e, all coordinates of $\mathbf{b}$ are positive, then the System (4.1.6) has a positive solution $\mathbf{x} \gg \mathbf{0}$. The existence of positive solutions of various classes of linear systems have been studied in [34, 56, 62].

The existence and uniqueness of positive solutions of the nonlinear algebraic system

$$A\mathbf{u} = \lambda g(\mathbf{u}) \tag{4.1.7}$$

have been investigated in [13, 70, 71, 73, 74, 75], where $A \in \mathbb{R}^{n\times n}$, $\mathbf{u} = (u_1, \ldots, u_n)^T \in \mathbb{R}^n$, $\lambda > 0$ and $f(\mathbf{u}) = (f_1(u_1), \ldots, f_n(u_n))^T$. It was demonstrated in [74] that positive solutions of such systems appear in several problems including finding positive solutions of a finite difference approximation of second-order differential equations with periodic boundary conditions, periodic solutions of fourth-order difference equations, second-order lattice dynamic systems, discrete neural networks.

If A is invertible, we can rewrite (4.1.7) as $\mathbf{u} = \lambda A^{-1} g(\mathbf{u})$. Then, assuming g is also invertible, using $f_i(u) = g_i^{-1}(u)$, and introducing the new variables $x_i = g_i(u_i)$, we get a nonlinear system of the form

$$f_i(x_i) = \sum_{j=1}^{n} c_{ij} x_j, \qquad 1 \leq i \leq n. \tag{4.1.8}$$

In many applications (see [76]) we have that A^{-1} is a positive matrix, i.e., all its coefficients are positive, hence we assume $c_{ij} > 0$ for all $i, j = 1, \ldots, n$. The existence and uniqueness of the positive solutions of the System (4.1.8) was investigated in [19, 76] for the special case $f_i(u) = u^\gamma$, and in [20] for the case when all the functions f_i are equal to a given function f.

Recently, in [21] the existence and uniqueness of positive solutions of the nonlinear system

$$f_i(x_i) = \sum_{j=1}^{n} c_{ij} x_j + p_i, \qquad 1 \leq i \leq n \tag{4.1.9}$$

was investigated using Brouwer's fixed point theorem under the conditions $c_{ij} > 0$ for all $i, j = 1, \ldots, n$ and $p_i \geq 0$.

The goal of this chapter is to study the existence and uniqueness of the positive solutions of the general nonlinear system

$$\gamma_i(x_i) = \sum_{j=1}^{n} g_{ij}(x_j), \qquad 1 \leq i \leq n. \tag{4.1.10}$$

Note that the System (4.1.10) includes the steady-state equations of a nonzero equilibrium of the dynamical systems (4.1.2), (4.1.4) and (4.1.5), respectively. Our main result, Theorem 4.2.1 below, uses a monotone iterative method to prove existence of a positive solution, and an extension of the method used in [21] to prove uniqueness under a weaker condition than that assumed in [21].

The structure of this chapter is the following. In Section 4.2 we formulate our main results. Theorem 4.2.1 below gives sufficient conditions to imply the existence and uniqueness of the positive solutions of the System (4.1.10). In Section 4.3

we show several examples including the Equations (4.1.6) and (4.1.9), where Theorem 4.2.1 is applicable.

4.2 Main results of Chapter 4

Consider the nonlinear system

$$\gamma_i(x_i) = \sum_{j=1}^{n} g_{ij}(x_j), \qquad 1 \le i \le n, \tag{4.2.1}$$

where $\gamma_i \in C(\mathbb{R}_+, \mathbb{R})$, $g_{ij} \in C(\mathbb{R}_+, \mathbb{R}_+)$, $1 \le i,j \le n$ and $\mathbb{R}_+ := [0, \infty)$. By a positive solution of the System (4.2.1) we mean a column vector $\mathbf{x} := (x_1, ..., x_n)^T$ which satisfies (4.2.1), and $x_1 > 0, ..., x_n > 0$.

We use the monotone iteration method in the proof of our main result, so we need the monotonicity of the functions γ_i, g_{ij} and the ratio $\frac{\gamma_j(u)}{g_{ij}(u)}$ which appear in the conditions of the next main theorem.

Next we formulate the main result of this chapter.

Theorem 4.2.1. *Let $\gamma_i : \mathbb{R}_+ \to \mathbb{R}$ and $g_{ij} : \mathbb{R}_+ \to \mathbb{R}_+$, $1 \le i,j \le n$ be continuous functions such that for each $1 \le i \le n$,*

(A) *there exists a $u_i^* > 0$ satisfying*

$$\gamma_i(u) \begin{cases} < 0, & \text{if} \quad 0 < u < u_i^*, \\ = 0, & \text{if} \quad u = u_i^*, \\ > 0, & \text{if} \quad u > u_i^*, \end{cases} \tag{4.2.2}$$

and γ_i is strictly increasing on $[u_i^*, \infty)$.

(B) *g_{ij}, $1 \le i,j \le n$ is increasing on $\mathbb{R}_+$, and there exists a $u_i^{**} \ge u_i^*$ such that*

$$\sum_{j=1}^{n} g_{ij}(u) < \gamma_i(u), \qquad u > u_i^{**}, \qquad 1 \le i \le n. \tag{4.2.3}$$

Then the System (4.2.1) has a positive solution.

Moreover, assume that

(C) *for each* $1 \le i,j \le n$, *either* $g_{ij}(u) > 0$ *for* $u > 0$ *or* $g_{ij}(u) = 0$ *for* $u > 0$, *i.e.,* g_{ij} *is either positive or constant* 0 *for* $u > 0$*;*

(D) *for each* $1 \le i,j \le n$, $\frac{\gamma_j(u)}{g_{ij}(u)}$ *is monotone increasing on* (u_j^*, ∞), *assuming* $g_{ij}(u) > 0$ *for* $u > 0$, *and there exist* i, j *such that* $g_{ij}(u) > 0$ *for* $u > 0$ *and* $\frac{\gamma_j(u)}{g_{ij}(u)}$ *is strictly monotone increasing on* (u_j^*, ∞).

Then the System (4.2.1) has a unique positive solution.

Proof. Let $B_i := \lim_{u\to\infty} \gamma_i(u)$, $i = 1, \dots, n$. Then either B_i is positive finite or it is ∞. Note that assumption (4.2.3) yields that $\sum_{j=1}^{n} g_{ij}(u) \le B_i$ for $u \ge 0$ and $i = 1, \dots, n$. Assumption **(A)** implies that, for each $i = 1, ..., n$, γ_i restricted to $[u_i^*, \infty)$ has an inverse, i.e., there exists a continuous strictly increasing function $h_i : [0, B_i) \to [u_i^*, \infty)$ satisfying

$$\gamma_i(h_i(u)) = u, \ u \in [0, B_i), \qquad h_i(\gamma_i(u)) = u, \ u \ge u_i^* \quad \text{and} \quad h_i(0) = u_i^*. \tag{4.2.4}$$

Now we have from (4.2.1) and the definition of h_i that (4.2.1) has a positive solution $(x_1, ..., x_n)^T$ if and only if

$$x_i = h_i\left(\sum_{j=1}^{n} g_{ij}(x_j)\right), \qquad 1 \le i \le n.$$

Fix any $\underline{u} > 0$ and $\overline{u} > 0$ such that

$$\underline{u} < \min_{1\le i\le n} u_i^* \le \max_{1\le i\le n} u_i^{**} < \overline{u}.$$

Then (4.2.3) and (4.2.4) yield

$$\underline{u} \le h_i\left(\sum_{j=1}^{n} g_{ij}(\underline{u})\right) \le h_i\left(\sum_{j=1}^{n} g_{ij}(\overline{u})\right) \le \overline{u}, \qquad 1 \le i \le n. \tag{4.2.5}$$

Now, for each $i = 1, ..., n$, we construct a sequence $(x_i^{(0)}, ..., x_i^{(k)}, ...)$ by the definition

$$x_i^{(0)} = \underline{u} \ \text{ and } \ x_i^{(k+1)} = h_i\left(\sum_{j=1}^{n} g_{ij}(x_j^{(k)})\right), \qquad k \ge 0, \tag{4.2.6}$$

and we prove that the sequence $(x_i^{(0)}, ..., x_i^{(k)}, ...)$ is convergent. For this aim, we prove that the sequence $(x_i^{(0)}, ..., x_i^{(k)}, ...)$ is monotone increasing and bounded from above. First we show, for each fixed $i = 1, ..., n$, that

$$x_i^{(k+1)} \geq x_i^{(k)}, \qquad \text{for all} \quad k \geq 0. \tag{4.2.7}$$

We use the mathematical induction. At $k = 0$ we have, by (4.2.5) and (4.2.6),

$$x_i^{(1)} = h_i\left(\sum_{j=1}^{n} g_{ij}(x_j^{(0)})\right) = h_i\left(\sum_{j=1}^{n} g_{ij}(\underline{u})\right) \geq \underline{u} = x_i^{(0)}.$$

Next, we assume that for some $k \geq 1$

$$x_i^{(k)} \geq x_i^{(k-1)}. \tag{4.2.8}$$

Then, by (4.2.6) and (4.2.8) and the monotonicity of g_{ij} and h_i, we have

$$x_i^{(k+1)} = h_i\left(\sum_{j=1}^{n} g_{ij}(x_j^{(k)})\right) \geq h_i\left(\sum_{j=1}^{n} g_{ij}(x_j^{(k-1)})\right) = x_i^{(k)}.$$

Hence the sequence $(x_i^{(0)}, ..., x_i^{(k)}, ...)$ is monotone increasing.

Now to prove that the sequence $(x_i^{(0)}, ..., x_i^{(k)}, ...)$ is bounded from above for all $1 \leq i \leq n$, we show that

$$x_i^{(k+1)} \leq \overline{u}, \qquad \text{for all} \quad k \geq 0, \qquad 1 \leq i \leq n. \tag{4.2.9}$$

Again we use the mathematical induction. So, for a fixed $i = 1, ..., n$, at $k = 0$ we have by (4.2.5) and (4.2.6) that

$$x_i^{(1)} = h_i\left(\sum_{j=1}^{n} g_{ij}(x_j^{(0)})\right) = h_i\left(\sum_{j=1}^{n} g_{ij}(\underline{u})\right) \leq h_i\left(\sum_{j=1}^{n} g_{ij}(\overline{u})\right) \leq \overline{u}.$$

Next, we assume for some $k \geq 0$ that

$$x_i^{(k)} \leq \overline{u}. \tag{4.2.10}$$

Then, by (4.2.5) and (4.2.10) and the monotonicity of g_{ij} and h_i, we have

$$x_i^{(k+1)} = h_i\left(\sum_{j=1}^{n} g_{ij}(x_j^{(k)})\right) \leq h_i\left(\sum_{j=1}^{n} g_{ij}(\overline{u})\right) \leq \overline{u},$$

and hence the sequence $(x_i^{(0)}, ..., x_i^{(k)}, ...)$ is bounded from above for all $1 \leq i \leq n$. Now since the sequence is monotone increasing and bounded from above, then it converges to a finite limit, i.e., there exist positive constants x_i such that

$$\lim_{k \to \infty} x_i^{(k)} = x_i, \qquad 1 \leq i \leq n.$$

On the other hand,

$$x_i = \lim_{k\to\infty} x_i^{(k+1)} = \lim_{k\to\infty} h_i\left(\sum_{j=1}^{n} g_{ij}(x_j^{(k)})\right) = h_i\left(\sum_{j=1}^{n} g_{ij}(x_j)\right), \qquad 1 \le i \le n,$$

and hence (4.2.1) has a positive solution.

Now, we show the uniqueness of the solution of the System (4.2.1). Suppose that $(u_1, ..., u_n)$ and $(v_1, ..., v_n)$ are two positive solutions of the System (4.2.1). Then for each $1 \le i \le n$, we have

$$\gamma_i(u_i) = \sum_{j=1}^{n} g_{ij}(u_j), \qquad \text{and} \qquad \gamma_i(v_i) = \sum_{j=1}^{n} g_{ij}(v_j). \tag{4.2.11}$$

Since

$$\gamma_i(u_i) = \sum_{j=1}^{n} g_{ij}(u_j) \ge 0, \qquad \text{and} \qquad \gamma_i(v_i) = \sum_{j=1}^{n} g_{ij}(v_j) \ge 0,$$

it follows from **(A)** that $u_i \ge u_i^*$ and $v_i \ge u_i^*$ for $i = 1, ..., n$. Let $H = \{(i, j) : 1 \le i, j \le n, g_{ij}(u) > 0 \text{ for } u > 0\}$. If the set H is empty, then (4.2.11) reduces to

$$\gamma_i(u_i) = 0, \qquad \text{and} \qquad \gamma_i(v_i) = 0,$$

and hence **(A)** implies that $u_i = u_i^* = v_i$ for $i = 1, ..., n$, and so the uniqueness is proved. Therefore, for the rest of the proof, we assume that $H \ne \emptyset$. Define $(l, s), (k, r) \in H$ such that

$$\frac{g_{ls}(u_s)}{g_{ls}(v_s)} \le \frac{g_{ij}(u_j)}{g_{ij}(v_j)} \le \frac{g_{kr}(u_r)}{g_{kr}(v_r)}, \qquad (i, j) \in H. \tag{4.2.12}$$

We consider two cases:

(i) Suppose first that

$$\frac{g_{ls}(u_s)}{g_{ls}(v_s)} = \frac{g_{kr}(u_r)}{g_{kr}(v_r)}.$$

Then (4.2.12) yields that there exists a $\lambda > 0$ such that $g_{ij}(u_j) = \lambda g_{ij}(v_j)$ for $(i, j) \in H$. But then $g_{ij}(u_j) = \lambda g_{ij}(v_j)$ for all $1 \le i, j \le n$. Therefore, from (4.2.11), we have

$$\gamma_i(u_i) - \lambda\gamma_i(v_i) = \sum_{j=1}^{n} [g_{ij}(u_j) - \lambda g_{ij}(v_j)] = 0, \qquad 1 \le i \le n.$$

It follows that

$$\frac{\gamma_j(u_j)}{\gamma_j(v_j)} = \lambda, \qquad 1 \le j \le n, \qquad \text{and} \qquad \lambda = \frac{g_{ij}(u_j)}{g_{ij}(v_j)}, \qquad (i,j) \in H,$$

which implies that

$$\frac{\gamma_j(u_j)}{g_{ij}(u_j)} = \frac{\gamma_j(v_j)}{g_{ij}(v_j)}, \qquad (i,j) \in H.$$

By our assumption, there exists $(\bar{i}, \bar{j}) \in H$ such that the function $\frac{\gamma_{\bar{j}}}{g_{\bar{i}\bar{j}}}$ is strictly monotone increasing. For such $\bar{i}$ and $\bar{j}$, we have that $u_{\bar{j}} = v_{\bar{j}}$ and thus $\lambda = 1$. Hence $\gamma_i(u_i) = \gamma_i(v_i),\ 1 \le i \le n$, which implies $u_i = v_i,\ 1 \le i \le n$. Therefore the solution of the System (4.2.1) is unique.

(ii) Suppose now that

$$\frac{g_{ls}(u_s)}{g_{ls}(v_s)} < \frac{g_{kr}(u_r)}{g_{kr}(v_r)}. \tag{4.2.13}$$

Note that (4.2.12) yields

$$g_{ij}(u_j)g_{ls}(v_s) - g_{ij}(v_j)g_{ls}(u_s) \ge 0, \qquad 1 \le i,j \le n, \tag{4.2.14}$$

and

$$g_{ij}(v_j)g_{kr}(u_r) - g_{ij}(u_j)g_{kr}(v_r) \ge 0, \qquad 1 \le i,j \le n. \tag{4.2.15}$$

With $i = s$, (4.2.11) implies

$$\gamma_s(u_s) = \sum_{j=1}^{n} g_{sj}(u_j), \qquad \text{and} \qquad \gamma_s(v_s) = \sum_{j=1}^{n} g_{sj}(v_j),$$

hence

$$\gamma_s(u_s)g_{ls}(v_s) - \gamma_s(v_s)g_{ls}(u_s) = \sum_{j=1}^{n} [g_{sj}(u_j)g_{ls}(v_s) - g_{sj}(v_j)g_{ls}(u_s)].$$

Using (4.2.14) and that $g_{ls}(u_s) > 0, g_{ls}(v_s) > 0$, we get

$$0 \le \gamma_s(u_s)g_{ls}(v_s) - \gamma_s(v_s)g_{ls}(u_s) = g_{ls}(u_s)g_{ls}(v_s)\left(\frac{\gamma_s(u_s)}{g_{ls}(u_s)} - \frac{\gamma_s(v_s)}{g_{ls}(v_s)}\right).$$

Since $\frac{\gamma_s(u)}{g_{ls}(u)}$ is monotone increasing, it follows $u_s \ge v_s$. Similarly, with $i = r$, (4.2.11) implies

$$\gamma_r(u_r)g_{kr}(v_r) - \gamma_r(v_r)g_{kr}(u_r) = \sum_{j=1}^{n}[g_{rj}(u_j)g_{kr}(v_r) - g_{rj}(v_j)g_{kr}(u_r)].$$

Using (4.2.15) and that $g_{kr}(u_r) > 0, g_{kr}(v_r) > 0$, we get

$$0 \geq \gamma_r(u_r)g_{kr}(v_r) - \gamma_r(v_r)g_{kr}(u_r) = g_{kr}(u_r)g_{kr}(v_r)\left(\frac{\gamma_r(u_r)}{g_{kr}(u_r)} - \frac{\gamma_r(v_r)}{g_{kr}(v_r)}\right).$$

Since $\frac{\gamma_r(u)}{g_{kr}(u)}$ is monotone increasing, we get $u_r \leq v_r$. The monotonicity of the functions g_{ij} implies that $g_{ls}(u_s) \geq g_{ls}(v_s)$ and $g_{kr}(u_r) \leq g_{kr}(v_r)$, and therefore $g_{ls}(v_s)g_{kr}(u_r) - g_{ls}(u_s)g_{kr}(v_r) \leq 0$, which contradicts with (4.2.13). Hence the System (4.2.1) has a unique solution, and the proof is completed. □

4.3 Applications

In this section we investigate special cases of the general System (4.2.1). We show several examples which demonstrate that Theorem 4.2.1 generalizes known existence and uniqueness results of the literature.

4.3.1 Nonlinear systems

Next we consider the nonlinear system

$$a_i x_i^{\alpha_i} = \sum_{j=1}^{n} c_{ij} x_j^{\beta_{ij}} + p_i, \qquad 1 \leq i \leq n. \tag{4.3.1}$$

If we set $\beta_{ij} = 1$ for all i, j, then the corresponding Equation (4.3.1) will be a special case of (4.1.8) with $f_i(u) = a_i u^{\alpha_i}$. For this case it was shown in [21] that if $a_i > 0$, $\alpha_i > 1$, $p_i \geq 0$, $\beta_{ij} = 1$ and $c_{ij} > 0$ for $1 \leq i, j \leq n$, then (4.3.1) has a unique positive solution. Now in the next result we show the existence and uniqueness of the solution of (4.3.1) under weaker assumption even in the above special case, since c_{ij} is allowed to be 0, and we suppose that one of the parameters c_{ii} or p_i is positive for all $i = 1, \ldots, n$.

Corollary 4.3.1. *Assume that $a_i > 0$, $p_i \geq 0$ and $c_{ij} \geq 0$ for each $1 \leq i, j \leq n$ are such that $c_{ii} + p_i > 0$ for $1 \leq i \leq n$. Then the System (4.3.1) has a unique positive solution assuming that $\alpha_i > \beta_{ij} \geq 0$ for all $1 \leq i, j \leq n$.*

Proof. Equation (4.3.1) can be written in the form (4.2.1) with $\gamma_i(u) := a_i u^{\alpha_i} - c_{ii}u^{\beta_{ii}} - p_i$, $g_{ij}(u) := c_{ij}u^{\beta_{ij}}$ for each $1 \leq i \neq j \leq n$ and $g_{ii}(u) = 0$. Now, we check that conditions **(A)** and **(B)** of Theorem 4.2.1 are satisfied. For condition **(A)**, we have $\gamma_i(u) = 0$, $1 \leq i \leq n$, if and only if

$$a_i u^{(\alpha_i - \beta_{ii})} = c_{ii} + \frac{p_i}{u^{\beta_{ii}}}, \qquad 1 \leq i \leq n. \tag{4.3.2}$$

It is clear that the left hand side of (4.3.2) is an increasing function and the right hand side of (4.3.2) is a decreasing function if and only if $\alpha_i > \beta_{ii} \geq 0$ for all $1 \leq i \leq n$. So it is easy to see, using the assumed conditions, that their graphs intersect in a unique point $u_i^* > 0$, therefore there exists a $u_i^* > 0$ which satisfies (4.2.2). Note that

$$\gamma_i'(u) = \alpha_i a_i u^{(\alpha_i - 1)} - c_{ii}\beta_{ii}u^{(\beta_{ii}-1)} = u^{(\beta_{ii}-1)}\left(\alpha_i a_i u^{(\alpha_i - \beta_{ii})} - c_{ii}\beta_{ii}\right) > 0,$$

if

$$u > \bar{u}_i := \left(\frac{c_{ii}\beta_{ii}}{a_i\alpha_i}\right)^{\frac{1}{\alpha_i - \beta_{ii}}} \geq 0, \qquad 1 \leq i \leq n.$$

Since $\gamma_i(\bar{u}_i) < 0$, we have $u_i^* > \bar{u}_i$, and therefore $\gamma_i(u)$ is strictly increasing on $[u_i^*, \infty)$ and condition **(A)** is satisfied. To check condition **(B)**, we see that $g_{ij}(u) := c_{ij}u^{\beta_{ij}}$, $1 \leq i \neq j \leq n$, and $g_{ii}(u) = 0$ are increasing on $\mathbb{R}_+$, and (4.2.3) is satisfied if and only if

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} c_{ij}u^{\beta_{ij}} < a_i u^{\alpha_i} - c_{ii}u^{\beta_{ii}} - p_i \Leftrightarrow \sum_{j=1}^{n} c_{ij}u^{(\beta_{ij} - \alpha_i)} < a_i - \frac{p_i}{u^{\alpha_i}},$$

therefore (4.2.3) is satisfied with a large enough u_i^{**}. Therefore (4.3.1) has a positive solution.

Now, we check conditions **(C)** and **(D)** of Theorem 4.2.1. Since $c_{ij} \geq 0$ for each $1 \leq i, j \leq n$, then condition **(C)** holds. If $c_{ij} = 0$ for all $1 \leq i, j \leq n$, then **(D)** is satisfied. Assuming that $c_{ij} > 0$ for some $1 \le i, j \le n$, then

$$\frac{\gamma_j(u)}{g_{ij}(u)} = \frac{a_j u^{\alpha_j} - c_{jj} u^{\beta_{jj}} - p_j}{c_{ij} u^{\beta_{ij}}} = \frac{a_j u^{(\alpha_j - \beta_{ij})}}{c_{ij}} - \frac{c_{jj}}{c_{ij}} u^{(\beta_{jj} - \beta_{ij})} - \frac{p_j}{c_{ij} u^{\beta_{ij}}}. \tag{4.3.3}$$

If $\beta_{jj} < \beta_{ij}$, then each term in (4.3.3) is strictly monotone increasing on $(0, \infty)$, and hence so is $\frac{\gamma_j(u)}{g_{ij}(u)}$. If $\beta_{jj} \ge \beta_{ij}$, then it follows from (4.3.3) that

$$\frac{\gamma_j(u)}{g_{ij}(u)} = \frac{u^{(\beta_{jj} - \beta_{ij})}}{c_{ij}} \left(a_j u^{(\alpha_j - \beta_{jj})} - c_{jj} \right) - \frac{p_j}{c_{ij} u^{\beta_{ij}}},$$

which is also strictly monotone increasing on $(0, \infty)$, so condition **(D)** is satisfied. Hence, by Theorem 4.2.1, the System (4.3.1) has a unique positive solution, and the proof is completed. □

Now we consider the system

$$f_i(x_i) = \sum_{j=1}^{n} c_{ij} x_j + p_i, \qquad 1 \le i \le n \tag{4.3.4}$$

which was studied in [21]. It was assumed in [21] that the function $\frac{f_i(u)}{u}$ is strictly increasing for all $i = 1, \ldots, n$, $c_{ij} > 0$ for all $1 \le i, j \le n$, and for every $i = 1, \ldots, n$ and $s_i = c_{i1} + \cdots + c_{in}$ there exists $t_i > 0$ such that $\frac{f_i(t_i)}{t_i} = s_i$. Then the System (4.3.4) has a unique positive solution. Our main result of Theorem 4.2.1 gives back this results under a weaker assumption that c_{ij} can take the values 0, and only either c_{ii} or p_i is assumed to be positive for all $i = 1, \ldots, n$.

Corollary 4.3.2. *Assume that, for each $i = 1, ..., n$, $f_i : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, such that $\frac{f_i(u)}{u}$ is strictly increasing, and*

$$\lim_{u \to 0^+} \frac{f_i(u)}{u} \begin{cases} < \infty, & \text{if} \quad p_i > 0, \\ = 0, & \text{if} \quad p_i = 0, \end{cases} \qquad \text{and} \qquad \lim_{u \to \infty} \frac{f_i(u)}{u} > \sum_{j=1}^{n} c_{ij}, \qquad i = 1, ..., n.$$

Furthermore, assume that $p_i \ge 0$ and $c_{ij} \ge 0$ for each $1 \le i, j \le n$ are such that $c_{ii} + p_i > 0$ for $1 \le i \le n$. Then the System (4.3.4) has a unique positive solution.

Proof. We can rewrite (4.3.4) in the form (4.2.1) with $\gamma_i(u) := f_i(u) - c_{ii} u - p_i$ and $g_{ij}(u) := c_{ij} u$ for each $1 \le i \ne j \le n$ and $g_{ii}(u) = 0$. Now, we check that conditions **(A)** and **(B)** of Theorem 4.2.1 are satisfied. For condition **(A)**, we have with $u_i^* > 0$ that $\gamma_i(u_i^*) = 0$, if

$$\frac{f_i(u_i^*)}{u_i^*} = \frac{p_i}{u_i^*} + c_{ii}, \qquad 1 \le i \le n. \tag{4.3.5}$$

It is clear that the left hand side of (4.3.5) is an increasing function and the right hand side of (4.3.5) is a decreasing function, so the assumed conditions yield that their graphs intersect in a unique point $u_i^* > 0$, therefore there exists a $u_i^* > 0$ satisfying (4.2.2). We have that

$$\gamma_i(u) = u\left[\frac{f_i(u)}{u} - c_{ii}\right] - p_i, \qquad 1 \le i \le n,$$

is strictly increasing on $(0, \infty)$, and hence condition **(A)** is satisfied. To check condition **(B)**, we see that $g_{ij}(u) := c_{ij}u$, $1 \le i \ne j \le n$, and $g_{ii}(u) = 0$ are increasing on $\mathbb{R}_+$, and (4.2.3) is satisfied if and only if

$$\sum_{\substack{j=1 \\ j \ne i}}^{n} c_{ij}u < f_i(u) - c_{ii}u - p_i \Leftrightarrow \sum_{j=1}^{n} c_{ij} < \frac{f_i(u)}{u} - \frac{p_i}{u},$$

therefore (4.2.3) is satisfied when u is large enough. Hence condition **(B)** holds. Therefore (4.3.4) has a positive solution.

For the proof of the uniqueness of the positive solution of the System (4.3.4), we check conditions **(C)** and **(D)** of Theorem 4.2.1. Since $c_{ij} \ge 0$ for each $1 \le i, j \le n$, condition **(C)** is satisfied. Assuming that $c_{ij} > 0$ for some $1 \le i, j \le n$, we get

$$\frac{\gamma_j(u)}{g_{ij}(u)} = \frac{f_j(u) - c_{jj}u - p_j}{c_{ij}u} = \frac{f_j(u)}{c_{ij}u} - \frac{c_{jj}}{c_{ij}} - \frac{p_j}{c_{ij}u}$$

is strictly increasing on $(0, \infty)$ and so condition **(D)** is satisfied. Hence the System (4.3.4) has a unique positive solution. □

Now, we consider a more general system of nonlinear algebraic equations

$$\gamma_i(x_i) = \sum_{j=1}^{n} c_{ij}\sigma_j(x_j), \qquad 1 \le i \le n. \tag{4.3.6}$$

The System (4.3.6) includes the steady-state equations of the donor-controlled compartmental system (4.1.2) and the Cohen–Grossberg neural network model (4.1.5).

Corollary 4.3.3. *Assume that $c_{ij} \ge 0$, for each $1 \le i, j \le n$, $\gamma_i : (0, \infty) \to (0, \infty)$ and $\sigma_i : (0, \infty) \to (0, \infty)$ are continuous and strictly increasing for $i = 1, ..., n$, such* *that*

($\boldsymbol{A}^*$) *the function* γ_i, $i = 1, ..., n$, *satisfies condition* **(A)** *of Theorem 4.2.1;*

($\boldsymbol{B}^*$) *the functions* γ_i *and* σ_j, $1 \leq i, j \leq n$ *satisfy* $\sum\limits_{j=1}^{n} c_{ij}\sigma_j(u) < \gamma_i(u)$ *for large enough* u.

Then the System (4.3.6) has a positive solution.

Furthermore, assume that $\frac{\gamma_i(u)}{\sigma_i(u)}$ *is continuous and strictly increasing on* $(0, \infty)$, *for all* $1 \leq i \leq n$. *Then the System (4.3.6) has a unique positive solution.*

Proof. Equation (4.3.6) can be written in the form (4.2.1) with $g_{ij}(u) := c_{ij}\sigma_j(u)$ for each $1 \leq i, j \leq n$. Assumptions ($\mathbf{A}^*$) and ($\mathbf{B}^*$) show that conditions **(A)** and **(B)** of Theorem 4.2.1 are satisfied. Therefore (4.3.6) has a positive solution.

Now, we show that the positive solution the System (4.3.6) is unique. Since $c_{ij} \geq 0$ for each $1 \leq i, j \leq n$, then we see that $g_{ij}(u) = c_{ij}\sigma_j(u) > 0$ for $u > 0$ if $c_{ij} > 0$ and $g_{ij}(u) = 0$ for $u > 0$ if $c_{ij} = 0$, and hence condition **(C)** of Theorem 4.2.1 is satisfied. Assuming that $c_{ij} > 0$ for some $1 \leq i, j \leq n$, then

$$\frac{\gamma_j(u)}{g_{ij}(u)} = \frac{\gamma_j(u)}{c_{ij}\sigma_j(u)} = \frac{1}{c_{ij}}\frac{\gamma_j(u)}{\sigma_j(u)}$$

is strictly increasing on $(0, \infty)$, and so condition **(D)** of Theorem 4.2.1 holds. Hence the System (4.3.4) has a unique positive solution and the proof is completed. □

4.3.2 Two dimensional systems

We consider the System (4.2.1) in the special case when $n = 2$:

$$\begin{aligned} \psi_1(x_1) &= g_{11}(x_1) + g_{12}(x_2), \\ \psi_2(x_2) &= g_{21}(x_1) + g_{22}(x_2). \end{aligned} \tag{4.3.7}$$

Introducing $\gamma_i(u) = \psi_i(u) - g_{ii}(u)$, $i = 1, 2$, we get the equivalent system

$$\begin{aligned} \gamma_1(x_1) &= g_{12}(x_2), \\ \gamma_2(x_2) &= g_{21}(x_1). \end{aligned} \tag{4.3.8}$$

The following result shows that in this two dimensional case we can reduce the study of existence and uniqueness of solutions of the System (4.3.8) to that of a scalar equation.

Corollary 4.3.4. *Assume that, for each* $1 \leq i, j \leq 2$, $\gamma_i, g_{ij} \in C(\mathbb{R}_+, \mathbb{R}_+)$, *such that*

($\boldsymbol{B}_1$) *the functions* γ_1 *and* γ_2 *satisfy condition* **(A)** *of Theorem 4.2.1;*

($\boldsymbol{B}_2$) *the functions* g_{12} *and* g_{21} *satisfy condition* **(B)** *of Theorem 4.2.1.*

Then

(i) *the System (4.3.8) has a positive solution;*

(ii) *the positive vector* (u_1, u_2) *is a solution of (4.3.8) if and only if* u_1 *and* u_2 *are the solutions of the scalar equations*

$$u = h_1(g_{12}(h_2(g_{21}(u)))) \tag{4.3.9}$$

and

$$u = h_2(g_{21}(h_1(g_{12}(u)))) \tag{4.3.10}$$

respectively, where h_1 *and* h_2 *are defined by (4.2.4);*

(iii) *the positive solution of System (4.3.8) is unique if at least one of the Equations (4.3.9) or (4.3.10) (or equivalently both of them) has only a unique positive solution.*

Proof. The proof of **(i)** is the consequence of Theorem 4.2.1. For the proof of **(ii)**, we see that the Equations (4.3.9) and (4.3.10) follow from System (4.3.8) using the inverse of the functions γ_i, $i = 1, 2$. For the proof of **(iii)** we consider the case when, e.g., x_1 is a unique solution of (4.3.9), then clearly $(x_1, h_2(g_{21}(h_1(g_{12}(x_1)))))$ is the unique solution of the System (4.3.8). $\square$

Example 4.3.1. As an example on the two dimensional case, we consider the system

$$\begin{aligned} 2x_1 - 1 &= x_2, \\ x_2 - 0.5 &= g_{21}(x_1), \end{aligned} \tag{4.3.11}$$

where

$$g_{21}(u) = \begin{cases} 0.5, & \text{if} \quad u \in [0,1], \\ 2u - 1.5, & \text{if} \quad u \in [1,2], \\ 2.5, & \text{if} \quad u \in [2,\infty). \end{cases}$$

Define $\gamma_1(u) = 2u - 1,\ \gamma_2(u) = u - 0.5,\ g_{12}(u) = u$. Then, clearly, we can see that condition **(A)** of Theorem 4.2.1 is satisfied with $u_1^* = 0.5$ and $u_2^* = 0.5$. Also, condition **(B)** of Theorem 4.2.1 holds for the System (4.3.11), and so the System (4.3.11) has a positive solution. Condition **(C)** of Theorem 4.2.1 holds too. We have, from the definition of γ_1 and γ_2, that

$$h_1(u) = \frac{u+1}{2}, \qquad u \in \mathbb{R}_+, \qquad \textit{and} \qquad h_2(u) = u + 0.5, \qquad u \in \mathbb{R}_+.$$

Then Equation (4.3.10) reduces to

$$u = h_2(g_{21}(h_1(g_{12}(u)))) = h_2\left(g_{21}\left(\frac{u+1}{2}\right)\right) = \begin{cases} h_2(0.5), & \text{if} \quad u \in [0,1], \\ h_2(u-0.5), & \text{if} \quad u \in [1,3], \\ h_2(2.5), & \text{if} \quad u \in [3,\infty), \end{cases}$$

or equivalently,

$$u = \begin{cases} 1, & \text{if} \quad u \in [0,1], \\ u, & \text{if} \quad u \in [1,3], \\ 3, & \text{if} \quad u \in [3,\infty). \end{cases}$$

This shows that (4.3.10) has infinitely many solutions, say, $u_2 = t,\ \ t \in [1,3]$, then $(\frac{t+1}{2}, t),\ \ t \in [1,3]$ is a solution of the System (4.3.11). On the other hand, we have

$$\frac{\gamma_1(u)}{g_{21}(u)} = \frac{2u-1}{2u-1.5} = 1 + \frac{0.5}{2u-1.5}, \qquad u \in [1,2],$$

which is decreasing on $[1, 2]$. Also, we have

$$\frac{\gamma_2(u)}{g_{12}(u)} = \frac{u - 0.5}{u} = 1 - \frac{0.5}{u}, \qquad u \in [1, 2],$$

which is increasing on $[1, 2]$. So condition **(D)** of Theorem 4.2.1 is not satisfied in this case. This shows that if condition **(D)** of Theorem 4.2.1 does not hold, we may loose the uniqueness.

Chapter 5

Boundedness of positive solutions of a system of nonlinear delay differential equations

In this chapter, we present sufficient conditions for the uniform permanence of the positive solutions of the system of nonlinear delay differential equations

$$\dot{x}_i(t) = \sum_{j=1}^{n}\sum_{\ell=1}^{n_0}\alpha_{ij\ell}(t)h_{ij}(x_j(t-\tau_{ij\ell}(t))) - r_i(t)f_i(x_i(t)) + \rho_i(t),\ t \geq 0,\ 1 \leq i \leq n.$$

The structure of this chapter is the following: Section 5.1 introduces a description of our system of nonlinear delay differential equations and some basic preliminaries. In Section 5.2 we formulate our main results Theorem 5.2.4 below gives estimates for the limit inferior and limit superior of the positive solutions of System (5.2.1). In Section 5.3 we show several corollaries, where Theorem 5.2.4 works in a good way. In Section 5.4 we introduce some applications of our main result to some population models. In Section 5.5 we give some examples with numerical simulations to illustrate our main results of this chapter.

5.1 Introduction

Nonlinear differential equations with delays frequently appear as model equations in physics, engineering, economics and biology. As we mentioned in Chapter 4 for some typical applications like compartmental systems and neural networks (see [39, 41, 50, 55]).

In [16] the existence, uniqueness and global stability of asymptotically periodic solutions of the bidirectional associative memory (BAM) network

$$\begin{aligned} \dot{x}_i(t) &= -a_i(t)x_i(t) + \sum_{j=1}^{k} p_{ji}(t)f_j(y_j(t-\tau_{ji})) + I_i(t), \quad i = 1,\ldots,n, \quad (5.1.1) \\ \dot{y}_j(t) &= -b_j(t)y_j(t) + \sum_{i=1}^{n} q_{ij}(t)f_i(x_j(t-\sigma_{ij})) + J_i(t), \quad j = 1,\ldots,k \quad (5.1.2) \end{aligned}$$

was examined.

In [28] the delay model

$$\begin{aligned} \dot{R}(t) &= f(T(t-\tau_3)) - d_1R(t) \quad (5.1.3) \\ \dot{L}(t) &= r_1R(t-\tau_1) - d_2L(t) \quad (5.1.4) \\ \dot{T}(t) &= r_2L(t-\tau_3) - d_3T(t) \quad (5.1.5) \end{aligned}$$

was considered for the control of the secretion of the hormone testosterone. Here $R(t)$, $L(t)$ and $T(t)$ are the concentrations of the gonadotropinreleasing hormone, luteinizing hormone and testosterone, respectively, r_1, r_2, d_1, d_2, d_3 are positive constants. Global stability of a positive equilibrium and oscillations of the solutions were investigated depending on the values of a parameter in the formula of the positive nonlinear function f.

In [8] the two-dimensional system

$$\begin{aligned} \dot{x}(t) &= r_1(t)\Big[f_1(y(t-\tau_1(t)) - x(t)\Big], \qquad t \geq 0 \quad (5.1.6) \\ \dot{y}(t) &= r_2(t)\Big[f_2(x(t-\tau_2(t)) - y(t)\Big], \qquad t \geq 0 \quad (5.1.7) \end{aligned}$$

was considered as a special case of a more general two-dimensional system of nonlinear delay equations with distributed delays. Sufficient conditions were given implying that the solutions of the System (5.1.6)-(5.1.7) are permanent, i.e., there exist positive constants a, A, b and B such that $a \le x(t) \le A$ and $b \le y(t) \le B$ hold for $t \ge 0$.

Populations are frequently modelled in heterogenous environments due to, e.g., different food-rich patches, different stages of a species according to age or size. In such models time delays appear naturally due to the time needed for species to disperse from one patch to another. We recall here the n-dimensional Nicholson's blowflies systems with patch structure

$$\dot{x}_i(t) = \sum_{\ell=1}^{n_0} \beta_{i\ell} x_i(t-\tau_{i\ell}) e^{-x_i(t-\tau_{i\ell})} + \sum_{j=1}^{n} a_{ij} x_j(t) - d_i x_i(t), \qquad 1 \le i \le n, \quad (5.1.8)$$

where $d_i > 0, \beta_{i\ell} \ge 0, a_{ij} \ge 0, \tau_{i\ell} \ge 0$ for $i, j = 1, \dots, n$, $\ell = 1, \dots, n_0$. Asymptotic behavior, permanence of the solutions was investigated, e.g., in, [6, 7, 33, 59]. For the scalar case, this model reduces to the famous Nicholson's blowflies equation introduced in [37] to model the Australian sheep-blowfly population.

The n-dimensional population model with patch structure

$$\begin{aligned} \dot{x}_i(t) &= \sum_{\ell=1}^{n_0} \frac{\lambda_{i\ell}(t) x_i(t-\tau_{i\ell}(t))}{1+\gamma_{i\ell}(t) x_i(t-\tau_{i\ell}(t))} + \sum_{\substack{j=1\\ j\ne i}}^{n} a_{ij}(t) x_j(t-\sigma_{ij}(t)) \\ &\quad -\mu_i(t) x_i(t) - \kappa_i(t) x_i^2(t), \qquad t \ge 0, \quad 1 \le i \le n \qquad (5.1.9) \end{aligned}$$

was introduced in [32], and the permanence of the positive solutions was investigated. Here all functions are nonnegative. It is a generalization of a scalar modified logistic equation with several delays introduced in [4].

Motivated by the above models, in this chapter we consider a system of nonlinear delay differential equations of the form

$$\dot{x}_i(t) = \sum_{j=1}^{n} \sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t) h_{ij}(x_j(t-\tau_{ij\ell}(t))) - r_i(t) f_i(x_i(t)) + \rho_i(t), \qquad t \ge 0, \ \ 1 \le i \le n \qquad (5.1.10)$$

with the initial condition

$$x_i(t) = \varphi_i(t), \qquad -\tau \le t \le 0, \ \ 1 \le i \le n \tag{5.1.11}$$

where, $\tau > 0$, is a positive constant, $\varphi_i \in C_+$, $h_{ij}, f_i, r_i, \alpha_{ij\ell}, \rho_i \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $\tau_{ij\ell} \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $0 \le \tau_{ij\ell}(t) \le \tau$ for $t \ge 0, 1 \le i, j \le n$ and $1 \le \ell \le n_0$. Our main Theorem 5.2.4 below shows that, under certain conditions, the solutions of the initial value problem (IVP) (5.1.10) and (5.1.11) is uniformly permanent, i.e., there exist positive constants $k_1, \ldots, k_n, K_1, \ldots, K_n$, such that for any initial functions $\varphi_i \in C_+$, $i = 1, \ldots, n$ the corresponding solution satisfies

$$0 < k_i \le \liminf_{t\to\infty} x_i(t) \le \limsup_{t\to\infty} x_i(t) \le K_i, \qquad 1 \le i \le n. \tag{5.1.12}$$

Moreover, the constants $k_1, \ldots, k_n$ and $K_1, \ldots, K_n$ are given explicitly, as unique positive solutions of associated nonlinear algebraic systems. As a consequence of the main result, we formulate conditions which imply that all the positive solutions converge to a constant limit (see Corollary 5.3.1 below). In Theorem 5.3.3, for nonlinear systems of the form

$$\dot{x}_i(t) = \sum_{j=1}^{n} \sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t) x_j(t - \tau_{ij\ell}(t)) - r_i(t) x_i^{q_i}(t) + \rho_i(t), \qquad t \ge 0, \qquad 1 \le i \le n, \tag{5.1.13}$$

we give sufficient conditions which imply that the positive solutions are asymptotically equivalent, i.e., the difference of any two positive solutions tends to 0 as the time goes to ∞.

This chapter extends the method introduced for the scalar case in Chapter 3 to the nonlinear delay system (5.1.10). A key element of the proof of our Theorem 5.2.4 is a result proved in Chapter 4, where sufficient conditions are formulated implying that a certain nonlinear algebraic system associated to (5.1.10) has a unique positive solution (see Lemma 5.2.3 below).

5.2 Main Results of Chapter 5

In this section, we give estimates for the limit inferior and limit superior of all positive solutions of the IVP

$$\dot{x}_i(t) = \sum_{j=1}^{n} \sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t) h_{ij}(x_j(t - \tau_{ij\ell}(t))) - r_i(t) f_i(x_i(t)) + \rho_i(t), \qquad t \geq 0,\ \ 1 \leq i \leq n \tag{5.2.1}$$

with the initial condition

$$x_i(t) = \varphi_i(t), \qquad -\tau \leq t \leq 0,\ \ 1 \leq i \leq n \tag{5.2.2}$$

where, $\tau > 0$, is a positive constant and $\varphi_i \in C_+,\ 1 \leq i \leq n$.

Now, we list our conditions

($\mathbf{A_0}$) $\tau_{ij\ell} \in C(\mathbb{R}_+, \mathbb{R}_+)$ are such that $0 \leq \tau_{ij\ell}(t) \leq \tau$ for $t \geq 0$, $1 \leq i, j \leq n$ and $1 \leq \ell \leq n_0$;

($\mathbf{A_1}$) $r_i \in C(\mathbb{R}_+, \mathbb{R}_+)$ are such that $r_i(t) > 0$ for $t > 0$, $1 \leq i \leq n$, and

$$\int_0^\infty r_i(s)\, ds = \infty, \qquad 1 \leq i \leq n; \tag{5.2.3}$$

($\mathbf{A_2}$) $\alpha_{ij\ell} \in C(\mathbb{R}_+, \mathbb{R}_+)$, for all $1 \leq i, j \leq n$ and $1 \leq \ell \leq n_0$ are such that

$$\sup_{t>0} \frac{\sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t)}{r_i(t)} < \infty, \qquad 1 \leq i, j \leq n; \tag{5.2.4}$$

($\mathbf{A_3}$) $f_i \in C(\mathbb{R}_+, \mathbb{R}_+)$, $1 \leq i \leq n$, are strictly increasing with $f_i(0) = 0$ and f_i are locally Lipschitz continuous;

($\mathbf{A_4}$) $h_{ij} \in C(\mathbb{R}_+, \mathbb{R}_+)$ are increasing, locally Lipschitz continuous, and $h_{ij}(u) > 0$ for $u > 0$ and $1 \leq i, j \leq n$;

($\mathbf{A_5}$) $\rho_i \in C(\mathbb{R}_+, \mathbb{R}_+)$ and for each $i = 1, \ldots, n$,

$$\text{either} \quad \liminf_{t\to\infty} \frac{\rho_i(t)}{r_i(t)} > 0 \quad \text{or} \quad \limsup_{u\to 0^+} \frac{f_i(u)}{h_{ii}(u)} < \liminf_{t\to\infty} \frac{\sum_{\ell=1}^{n_0} \alpha_{ii\ell}(t)}{r_i(t)}, \tag{5.2.5}$$

$$\sup_{t>0} \frac{\rho_i(t)}{r_i(t)} < \infty, \qquad \lim_{u\to\infty} f_i(u) = \infty, \tag{5.2.6}$$

and

$$\sum_{j=1}^{n} \Bigg(\limsup_{t\to\infty} \frac{\sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t)}{r_i(t)}\Bigg) \limsup_{u\to\infty} \frac{h_{ij}(u)}{f_i(u)} < 1; \tag{5.2.7}$$

($\mathbf{A_6}$) (i) $\frac{f_i(u)}{h_{ij}(u)}$ is increasing and $\frac{h_{jj}(u)}{h_{ij}(u)}$ is decreasing on $(0,\infty)$, for each $1 \le i,j \le n$;

(ii) for each $1 \le i \le n$, either $\frac{f_i(u)}{h_{ii}(u)}$ is strictly increasing on the interval $(0,\infty)$ or $\Big(\liminf_{t\to\infty} \frac{\rho_i(t)}{r_i(t)} > 0$ and $h_{ii}(u)$ is strictly increasing on $(0,\infty)\Big)$;

(iii) either $\liminf_{t\to\infty} \frac{\sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t)}{r_i(t)} = 0$ for all $i,j \in \{1,\dots,n\}$ satisfying $i \ne j$; or there exist $i,j \in \{1,\dots,n\}$, $i \ne j$ such that $\liminf_{t\to\infty} \frac{\sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t)}{r_i(t)} > 0$ and $\Big[$either $\frac{f_j(u)}{h_{ij}(u)}$ is strictly increasing on $(0,\infty)$ or $\Big(\liminf_{t\to\infty} \frac{\sum_{\ell=1}^{n_0} \alpha_{jj\ell}(t)}{r_j(t)} > 0$ and $\frac{h_{jj}(u)}{h_{ij}(u)}$ is strictly decreasing on $(0,\infty)\Big)$ or $\Big(\liminf_{t\to\infty} \frac{\rho_j(t)}{r_j(t)} > 0$ and $h_{ij}(u)$ is strictly increasing on $(0,\infty)\Big)\Big]$;

(iv) for each $1 \le i \le n$, either $\frac{f_i(u)}{h_{ii}(u)}$ is strictly increasing on the interval $(0,\infty)$ or $\Big(\limsup_{t\to\infty} \frac{\rho_i(t)}{r_i(t)} > 0$ and $h_{ii}(u)$ is strictly increasing on $(0,\infty)\Big)$;

(v) either $\limsup_{t\to\infty} \frac{\sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t)}{r_i(t)} = 0$ for all $i,j \in \{1,\dots,n\}$ satisfying $i \ne j$; or there exist $i,j \in \{1,\dots,n\}$, $i \ne j$ such that $\limsup_{t\to\infty} \frac{\sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t)}{r_i(t)} > 0$ and $\Big[$either $\frac{f_j(u)}{h_{ij}(u)}$ is strictly increasing on $(0,\infty)$ or $\Big(\limsup_{t\to\infty} \frac{\sum_{\ell=1}^{n_0} \alpha_{jj\ell}(t)}{r_j(t)} > 0$ and $\frac{h_{jj}(u)}{h_{ij}(u)}$ is strictly decreasing on $(0,\infty)\Big)$ or $\Big(\limsup_{t\to\infty} \frac{\rho_j(t)}{r_j(t)} > 0$ and $h_{ij}(u)$ is strictly increasing on $(0,\infty)\Big)\Big]$.

The boundedness of the delay functions is assumed throughout the chapter. Assumption relation (5.2.4) in ($\mathbf{A_2}$) is natural in view of Section 2.2. In the proof we will factor out r_i from the right hand side of (5.2.1), so the boundedness and positivity of the fractions $\frac{\sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t)}{r_i(t)}$ and $\frac{\rho_i(t)}{r_i(t)}$ in ($\mathbf{A_2}$) and ($\mathbf{A_5}$) will be a natural condition later. The proof uses a monotone iteration technique, so the monotonicity of f_i and h_{ij} in ($\mathbf{A_3}$) and ($\mathbf{A_4}$) will be essential. We will use Theorem 4.2.1, so the monotonicity of the fractions $\frac{f_i}{h_{ij}}$ and $\frac{h_{jj}}{h_f ij}$ in ($\mathbf{A_6}$) is needed later, as well as the strict monotonicity any of the functions listed in ($\mathbf{A_6}$).

Clearly, under conditions ($\mathbf{A_1}$)-($\mathbf{A_5}$), the IVP (5.2.1) and (5.2.2) has a unique solution corresponding to any $\varphi = (\varphi_1, ..., \varphi_n) \in C_+^n$. This solution is denoted by $x(\varphi) = (x_1(\varphi), ..., x_n(\varphi))$. Note that in Chapter 3 a scalar version of (5.2.1) was studied where, instead of the local Lipschitz-continuity, it was assumed that f_i is such that for any nonnegative constants ϱ and L satisfying $L \neq \varrho$, one has

$$\int_L^{\varrho} \frac{ds}{f_i(\varrho) - f_i(s)} = +\infty. \tag{5.2.8}$$

Hence the solution studied in Chapter 3 was not necessary unique. It is easy to see that the locally Lipschitz-continuity of f_i implies condition (5.2.8). We assume the locally Lipschitz-continuity of f_i and h_{ij} to simplify the presentation, but it can be omitted as in Chapter 3.

We note that assumption ($\mathbf{A_3}$) is weaker than that used in the [4, 31], where, investigating permanence of a scalar population model, it was assumed that the coefficient function β_i is bounded below and above by positive constants.

The monotonicity assumptions of ($\mathbf{A_6}$) for the ratios $\frac{f_i(u)}{h_{ij}(u)}$ and $\frac{h_{jj}(u)}{h_{ij}(u)}$ are crutial for using Lemma 5.2.3 below. This assumption allows us to include examples when some ratios are constants, and only some of these functions are strictly monotone. This week form of the condition will be important when we apply our main results to the population models (5.1.8) and (5.1.9) (see Corollary 5.4.1 and 5.4.2 below).

First, we present the next Lemma which shows that all solutions of the System (5.2.1) corresponding to any initial function $\varphi = (\varphi_1, \varphi_2, ..., \varphi_n) \in C_+^n$ are positive on $\mathbb{R}_+$.

Lemma 5.2.1. *Assume that $\tau_{ij\ell}$ satisfies condition ($\boldsymbol{A_0}$), r_i satisfies condition ($\boldsymbol{A_1}$), f_i satisfies condition ($\boldsymbol{A_3}$) and $\alpha_{ij}, h_{ij}, \rho_i \in C(\mathbb{R}_+, \mathbb{R}_+)$, $1 \le i, j \le n$ and $1 \le \ell \le n_0$. Then for any $\varphi = (\varphi_1, \varphi_2, ..., \varphi_n) \in C_+^n$, the solution $x(t) = x(\varphi)(t) = (x_1(\varphi)(t), ..., x_n(\varphi)(t))$ of the IVP (5.2.1) and (5.2.2) obeys $x_i(t) > 0$ for $t \ge 0$, $1 \le i \le n$.*

Proof. Since $x_i(0) = \varphi_i(0) > 0$, $1 \le i \le n$, then if $x_i(t) > 0$ for $t \ge 0$, $1 \le i \le n$ then we are done. Otherwise at least one of $x_1(t), ..., x_n(t)$ is equal to zero for some positive t. Since the functions $x_1(t), ..., x_n(t)$ are continuous, then in the last case there exists a $t_1 \in (0, \infty)$ such that $x_i(t) > 0$ for $0 \le t < t_1$, $1 \le i \le n$ and $\min\{x_1(t_1), ..., x_n(t_1)\} = 0$. Since $\alpha_{ij\ell}(t) \ge 0, \tau_{ij\ell}(t) \ge 0$ $\rho_i(t) \ge 0$, $t \ge 0$, $1 \le i, j \le n$, $1 \le \ell \le n_0$, and $h_{ij}(u) \ge 0$, $u \ge 0$, $1 \le i, j \le n$, then from (5.2.1) we have

$$\dot{x}_i(t) \ge -r_i(t) f_i(x_i(t)), \qquad 1 \le i \le n, \qquad 0 \le t \le t_1.$$

But from Theorem 2.1.2 we have

$$x_i(t) \ge y_i(t), \qquad 1 \le i \le n, \quad 0 \le t \le t_1,$$

where $y_i(t) = y(0, \varphi_i(0), 0, r_i, f_i)(t)$, $1 \le i \le n$ is the unique positive solution of the differential equation

$$\dot{y}(t) = r_i(t)\Big(c - f_i(y(t))\Big), \qquad t \ge 0, \tag{5.2.9}$$

with $c = 0$ and with the initial condition

$$y_i(0) = x_i(0) = \varphi_i(0) > 0, \qquad 1 \le i \le n.$$

Lemma 3.2.1 yields $y_i(t) > 0$, for all $t \ge 0$. Then at $t = t_1$ we get $x_i(t_1) \ge y_i(t_1) > 0$, $1 \le i \le n$, which is a contradiction with our assumption that $\min\{x_1(t_1), ..., x_n(t_1)\} = 0$. Hence $x_i(t) > 0$, $1 \le i \le n$ for $t \in [0, \infty)$. $\square$

Lemma 5.2.2. *Assume that conditions $(\boldsymbol{A_0})$– $(\boldsymbol{A_5})$ are satisfied. Then for any $\varphi \in C_+^n$, the solution $x(t) = x(\varphi)(t) = (x_1(\varphi)(t), ..., x_n(\varphi)(t))$ of the IVP (5.2.1) and (5.2.2) satisfies*

$$0 < \inf_{t\geq 0} x_i(\varphi)(t) \leq \sup_{t\geq 0} x_i(\varphi)(t) < \infty, \qquad 1 \leq i \leq n. \tag{5.2.10}$$

Proof. First we show that

$$\inf_{t\geq 0} x_i(t) > 0, \qquad 1 \leq i \leq n. \tag{5.2.11}$$

Let $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n) \in C_+^n$ be an arbitrary fixed initial function. Then, by Lemma 5.2.1, the solution $x(t) = x(\varphi)(t) = (x_1(\varphi)(t), \ldots, x_n(\varphi)(t))$ obeys $x_i(t) > 0$, $1 \leq i \leq n$, $t \geq 0$. We claim that there exist $T > 0$ and $c > 0$ such that the following inequalities are satisfied, for every $i = 1, \ldots, n$,

$$\min_{0\leq t\leq T} x_i(t) > c \quad \text{and} \quad \left(\frac{\sum_{j=1}^{n}\sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t) h_{ij}(c)}{r_i(t)} + \frac{\rho_i(t)}{r_i(t)} \right) > f_i(c), \qquad t \geq T. \tag{5.2.12}$$

From (5.2.5), we have two cases:

(i) if i is such that $\liminf_{t\to\infty} \frac{\rho_i(t)}{r_i(t)} > 0$, then fix a $\xi_i > 0$ such that

$$\liminf_{t\to\infty} \frac{\rho_i(t)}{r_i(t)} > \xi_i > 0.$$

Thus there exists $T_i > 0$ such that

$$\frac{\rho_i(t)}{r_i(t)} > \xi_i > 0, \quad \text{for} \quad t \geq T_i.$$

Lemma 5.2.1 and $(\mathbf{A_3})$ imply that there exists a $c_i > 0$ such that

$$\min_{0\leq t\leq T} x_i(t) > c_i \quad \text{and} \quad f_i(u) < \xi_i \quad \text{for} \quad 0 < u \leq c_i.$$

Therefore (5.2.12) is satisfied for such i.

(ii) if i is such that $\limsup_{u\to 0^+} \frac{f_i(u)}{h_{ii}(u)} < \liminf_{t\to\infty} \frac{\sum_{\ell=1}^{n_0} \alpha_{ii\ell}(t)}{r_i(t)}$, then let $K_i > 0$ be such that

$$\limsup_{u\to 0^+} \frac{f_i(u)}{h_{ii}(u)} < K_i < \liminf_{t\to\infty} \frac{\sum_{\ell=1}^{n_0} \alpha_{ii\ell}(t)}{r_i(t)}.$$

Thus there exists $T_i > 0$ such that

$$K_i < \frac{\sum_{\ell=1}^{n_0} \alpha_{ii\ell}(t)}{r_i(t)}, \qquad t \geq T_i.$$

Also, there exists $c_i > 0$ such that

$$\frac{f_i(u)}{h_{ii}(u)} < K_i, \quad \text{for} \quad 0 < u \leq c_i \quad \text{and} \quad \min_{0 \leq t \leq T} x_i(t) > c_i.$$

Then we have

$$\frac{\sum_{j=1}^{n} \sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t) h_{ij}(c)}{r_i(t)} \geq \frac{1}{K_i} f_i(u) \frac{\sum_{\ell=1}^{n_0} \alpha_{ii\ell}(t)}{r_i(t)} > f_i(u), \qquad t \geq T_i, \quad 0 < u \leq c_i,$$

and hence (5.2.12) holds for such i. Therefore (5.2.12) is satisfied, for all $i = 1, \ldots, n$, with $T = \max\{T_1, \ldots, T_n\}$ and $c = \min\{c_1, \ldots, c_n\}$.

Now, in virtue of (5.2.12), either $x_i(t) > c$ for all $t \geq 0$, $1 \leq i \leq n$, or there exists $t_2 \in (T, \infty)$ such that $\min\{x_1(t_2), \ldots, x_n(t_2)\} = c$ and $x_i(t) > c$ for $t \in [0, t_2)$, $1 \leq i \leq n$. In this case at least one of the values of $x_1(t_2), \ldots, x_n(t_2)$ is equal to c. Assume, e.g., that $x_1(t_2) = c$, then $\dot{x}_1(t_2) \leq 0$. On the other hand, the monotonicity of h_{1j} and (5.2.12) yield that

$$\begin{aligned} \dot{x}_1(t_2) &= r_1(t_2) \left[\frac{\sum_{j=1}^{n} \sum_{\ell=1}^{n_0} \alpha_{1j\ell}(t_2) h_{1j}(x_j(t_2 - \tau_{1j\ell}(t_2)))}{r_1(t_2)} - f_1(x_1(t_2)) + \frac{\rho_1(t_2)}{r_1(t_2)} \right] \\ &\geq r_1(t_2) \left[\frac{\sum_{j=1}^{n} \sum_{\ell=1}^{n_0} \alpha_{1j\ell}(t_2) h_{1j}(c)}{r_1(t_2)} + \frac{\rho_1(t_2)}{r_1(t_2)} - f_1(c) \right] \\ &> 0, \end{aligned}$$

which is a contradiction, since $\dot{x}_1(t_2) \leq 0$. Hence $x_1(t) > c$ for all $t \geq 0$. Similarly, we can show that $x_i(t) > c$, for all $t \geq 0$, $2 \leq i \leq n$, and therefore (5.2.11) holds.

Now we show that

$$\sup_{t \geq 0} x_i(t) < \infty, \qquad 1 \leq i \leq n. \tag{5.2.13}$$

We claim that there exist $T > 0$ and $M > 0$ such that the following inequalities are satisfied, for every $i = 1, \dots, n$,

$$\max_{0 \le t \le T} x_i(t) < M \quad \text{and} \quad \left(\frac{\sum_{j=1}^{n} \sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t) h_{ij}(M)}{r_i(t)} + \frac{\rho_i(t)}{r_i(t)} \right) < f_i(M), \qquad t \ge T. \tag{5.2.14}$$

The second relation of (5.2.14) holds if

$$\left(\frac{\sum_{j=1}^{n} \sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t) \frac{h_{ij}(M)}{f_i(M)}}{r_i(t)} + \frac{1}{f_i(M)} \frac{\rho_i(t)}{r_i(t)} \right) < 1, \qquad t \ge T. \tag{5.2.15}$$

Using (5.2.7), there exists a $\mu_i > 0$ such that

$$\sum_{j=1}^{n} \Big(\limsup_{t \to \infty} \frac{\sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t)}{r_i(t)} \Big) \limsup_{u \to \infty} \frac{h_{ij}(u)}{f_i(u)} < \mu_i < 1,$$

then there exists an $\delta > 0$ such that

$$\sum_{j=1}^{n} \Big(\limsup_{t \to \infty} \frac{\sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t)}{r_i(t)} + \delta \Big) \Big(\limsup_{u \to \infty} \frac{h_{ij}(u)}{f_i(u)} + \delta \Big) < \mu_i.$$

Thus there exist $T_i > 0$ and $V_{1i} > 0$ such that

$$\sum_{j=1}^{n} \Big(\sup_{t \ge T_i} \frac{\sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t)}{r_i(t)} \Big) \frac{h_{ij}(u)}{f_i(u)} < \mu_i, \quad u \ge V_{1i}.$$

Moreover, there exists a $V_{2i} > 0$ such that

$$\frac{1}{f_i(u)} \sup_{t \ge T_i} \frac{\rho_i(t)}{r_i(t)} < 1 - \mu_i, \quad u \ge V_{2i},$$

and so there exists a large $M > 0$ such that (5.2.15) holds and $\max_{0 \le t \le T} x_i(t) < M$, with $T = \max\{T_1, \dots, T_n\}$, for all $1 = 1, \dots, n$. Hence inequality (5.2.14) is satisfied for each $i = 1, \dots, n$. Now, in virtue of (5.2.14), either $x_i(t) < M$ for all $t \ge 0$, $1 \le i \le n$, or there exists $t_3 \in (T, \infty)$ such that $\max\{x_1(t_3), \dots, x_n(t_3)\} = M$, and $x_i(t) < M$ for $t \in [0, t_3)$ and $i = 1, \dots, n$. In this case at least one of the values of $x_1(t_3), \dots, x_n(t_3)$ is equal to M. Assume, e.g., that $x_1(t_3) = M$, then $\dot{x}_1(t_3) \ge 0$.

On the other hand, using (5.2.14) and the monotonicity of h_{1j}, we have

$$\begin{aligned}
\dot{x}_1(t_3) &= r_1(t_3)\left[\frac{\sum\limits_{j=1}^{n}\sum\limits_{\ell=1}^{n_0}\alpha_{1j\ell}(t_3)h_{1j}(x_j(t_3-\tau_{1j\ell}(t_3)))}{r_1(t_3)} - f_1(x_1(t_3)) + \frac{\rho_1(t_3)}{r_1(t_3)}\right] \\
&\le r_1(t_3)\left[\frac{\sum\limits_{j=1}^{n}\sum\limits_{\ell=1}^{n_0}\alpha_{1j\ell}(t_3)h_{1j}(M)}{r_1(t_3)} - f_1(M) + \frac{\rho_1(t_3)}{r_1(t_3)}\right] \\
&< 0,
\end{aligned}$$

which is a contradiction, since $\dot{x}_1(t_3) \ge 0$. Hence $x_1(t) < M$, for all $t \ge 0$. Similarly, we can show that $x_i(t) < M$, for all $t \ge 0$, $2 \le i \le n$, and therefore we can see that (5.2.13) holds. $\square$

The next Lemma displays many properties of the positive solutions of the algebraic system

$$f_i(x_i) = \sum_{j=1}^{n} m_{ij}h_{ij}(x_j) + l_i, \qquad 1 \le i \le n. \tag{5.2.16}$$

We say that $x = (x_1, ..., x_n)$ is a positive solution of (5.2.16) if $x_i > 0$ for $i = 1, ..., n$.

Lemma 5.2.3. *Assume that $m_{ij} \ge 0$, $l_i \ge 0$ for $1 \le i, j \le n$, f_i satisfies condition ($\boldsymbol{A_3}$) and h_{ij} satisfies condition ($\boldsymbol{A_4}$). Suppose that*

($\boldsymbol{C_1}$) *$\frac{f_i(u)}{h_{ij}(u)}$ is increasing and $\frac{h_{jj}(u)}{h_{ij}(u)}$ is decreasing on $(0, \infty)$, for each $1 \le i, j \le n$;*

($\boldsymbol{C_2}$) *for each $1 \le i \le n$, either $\frac{f_i(u)}{h_{ii}(u)}$ is strictly increasing on $(0, \infty)$ or $\Big(l_i > 0$ and $h_{ii}(u)$ is strictly increasing on $(0, \infty)\Big)$;*

($\boldsymbol{C_3}$) *either $m_{ij} = 0$ for all $i, j \in \{1, \ldots, n\}$ satisfying $i \ne j$; or there exist $i, j \in \{1, \ldots, n\}$, $i \ne j$ such that $m_{ij} > 0$ and $\Big[$either $\frac{f_j(u)}{h_{ij}(u)}$ is strictly increasing on $(0, \infty)$ or $\Big(m_{jj} > 0$ and $\frac{h_{jj}(u)}{h_{ij}(u)}$ is strictly decreasing on $(0, \infty)\Big)$ or $\Big(l_j > 0$ and $h_{ij}(u)$ is strictly increasing on $(0, \infty)\Big)\Big]$;*

(***C*$_4$**) *the functions f_i and h_{ii} satisfy*

$$\textit{either} \quad l_i > 0, \quad \textit{or} \quad \lim_{u \to 0^+} \frac{f_i(u)}{h_{ii}(u)} < m_{ii}, \qquad 1 \le i \le n, \tag{5.2.17}$$

and

$$\sum_{j=1}^{n} m_{ij} \lim_{u\to\infty} \frac{h_{ij}(u)}{f_i(u)} < 1, \quad \textit{and} \quad \lim_{u\to\infty} f_i(u) = \infty \qquad 1 \le i \le n. \tag{5.2.18}$$

Then

(i) *the System (5.2.16) has a unique positive solution $x^* = (x_1^*, ..., x_n^*)$.*

(ii) *For any $x = (x_1, ..., x_n)$ satisfying*

$$x_i > 0, \qquad f_i(x_i) \ge \sum_{j=1}^{n} m_{ij} h_{ij}(x_j) + l_i, \qquad 1 \le i \le n, \tag{5.2.19}$$

one has

$$x_i \ge x_i^*, \qquad 1 \le i \le n. \tag{5.2.20}$$

(iii) *For any $x = (x_1, ..., x_n)$ satisfying*

$$x_i > 0, \qquad f_i(x_i) \le \sum_{j=1}^{n} m_{ij} h_{ij}(x_j) + l_i, \qquad 1 \le i \le n, \tag{5.2.21}$$

one has

$$x_i \le x_i^*, \qquad 1 \le i \le n. \tag{5.2.22}$$

Proof. See Appendix A. □

We use the following notations in our main theorem:

$$\underline{m}_{ij} := \liminf_{t\to\infty} \frac{\sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t)}{r_i(t)}, \qquad \overline{m}_{ij} := \limsup_{t\to\infty} \frac{\sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t)}{r_i(t)}, \qquad 1 \le i, j \le n, \tag{5.2.23}$$

$$\underline{l}_i := \liminf_{t\to\infty} \frac{\rho_i(t)}{r_i(t)}, \qquad \bar{l}_i := \limsup_{t\to\infty} \frac{\rho_i(t)}{r_i(t)}, \qquad 1 \le i \le n. \tag{5.2.24}$$

We note that ($\mathbf{A_2}$), ($\mathbf{A_5}$) and Lemma 5.2.2 yield $0 \le \underline{m}_{ij} < \infty$, $0 \le \overline{m}_{ij} < \infty$, $0 \le \underline{l}_i < \infty$, $0 \le \bar{l}_i < \infty$ for $1 \le i, j \le n$, and

$$0 < \liminf_{t\to\infty} x_i(t) \le \limsup_{t\to\infty} x_i(t) < \infty, \qquad 1 \le i \le n.$$

Now, we are ready to formulate the main result of this chapter.

Theorem 5.2.4. *Assume that conditions* $(\boldsymbol{A_0})$*–*$(\boldsymbol{A_5})$ *are satisfied.*

(i) If, in addition, $(\boldsymbol{A_6})$ *(i), (ii) and (iii) hold, then for any initial function* $\varphi = (\varphi_1, \ldots, \varphi_n) \in C_+^n$*, the solution* $x(\varphi)(t) = (x_1(\varphi)(t), \ldots, x_n(\varphi)(t))$ *of the IVP (5.2.1) and (5.2.2) obeys*

$$\underline{x}_i^* \leq \liminf_{t\to\infty} x_i(\varphi)(t), \qquad 1 \leq i \leq n,$$

where $(\underline{x}_1^*, \ldots, \underline{x}_n^*)$ *is the unique positive solution of the algebraic system*

$$f_i(x_i) = \sum_{j=1}^{n} \underline{m}_{ij} h_{ij}(x_j) + \underline{l}_i, \qquad 1 \leq i \leq n. \tag{5.2.25}$$

(ii) If, in addition, $(\boldsymbol{A_6})$ *(i), (iv) and (v) hold, then for any initial function* $\varphi = (\varphi_1, \ldots, \varphi_n) \in C_+^n$*, the solution* $x(\varphi)(t) = (x_1(\varphi)(t), \ldots, x_n(\varphi)(t))$ *of the IVP (5.2.1) and (5.2.2) obeys*

$$\limsup_{t\to\infty} x_i(\varphi)(t) \leq \overline{x}_i^*, \qquad 1 \leq i \leq n,$$

where $(\overline{x}_1^*, \ldots, \overline{x}_n^*)$ *is the unique positive solution of the algebraic system*

$$f_i(x_i) = \sum_{j=1}^{n} \overline{m}_{ij} h_{ij}(x_j) + \overline{l}_i, \qquad 1 \leq i \leq n. \tag{5.2.26}$$

Proof. See Appendix A. □

5.3 Corollaries

In this section, we introduce some corollaries which confirm the applicability of our main results.

Corollary 5.3.1. *Assume that conditions* $(\boldsymbol{A_0})$*–*$(\boldsymbol{A_6})$ *are satisfied, moreover, the finite limits*

$$m_{ij} := \lim_{t\to\infty} \frac{\sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t)}{r_i(t)} \quad \text{and} \quad l_i := \lim_{t\to\infty} \frac{\rho_i(t)}{r_i(t)}, \qquad 1 \leq i, j \leq n, \tag{5.3.1}$$

exist. Then, for any initial function $\varphi = (\varphi_1, \varphi_2, ..., \varphi_n) \in C_+^n$ *, the solutions* $x(t) = x(\varphi)(t) = (x_1(\varphi)(t), ..., x_n(\varphi)(t))$ *of the IVP (5.2.1) and (5.2.2) satisfy*

$$\lim_{t\to\infty} x_i(\varphi)(t) = x_i^*, \qquad 1 \le i \le n, \tag{5.3.2}$$

where $(x_1^*, \ldots, x_n^*)$ *is the unique positive solution of the system*

$$f_i(x_i) = \sum_{j=1}^{n} m_{ij} h_{ij}(x_j) + l_i, \qquad 1 \le i \le n. \tag{5.3.3}$$

Now, we study a special form of (5.2.1), consider the IVP

$$\dot{x}_i(t) = \sum_{j=1}^{n}\sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t) x_j^{p_{ij}}(t - \tau_{ij\ell}(t)) - r_i(t) x_i^{q_i}(t) + \rho_i(t), \qquad t \ge 0, \qquad 1 \le i \le n, \tag{5.3.4}$$

with the initial condition

$$x_i(t) = \varphi_i(t), \qquad -\tau \le t \le 0, \qquad 1 \le i \le n, \tag{5.3.5}$$

where $\tau > 0$, $\varphi = (\varphi_1, \varphi_2, ..., \varphi_n) \in C_+^n$ and $\alpha_{ij\ell}, r_i, \tau_{ij\ell} \in C(\mathbb{R}_+, \mathbb{R}_+)$, $p_{ij}, q_i \in \mathbb{R}_+$ for $1 \le i, j \le n$ and $1 \le \ell \le n_0$.

We remark that $(\mathbf{A_3})$, $(\mathbf{A_4})$, $(\mathbf{A_5})$ and $(\mathbf{A_6})$ hold if

$$q_i > p_{ij} \ge 1, \quad \text{and} \quad p_{ij} \ge p_{jj}, \qquad 1 \le i, j \le n \tag{5.3.6}$$

and

$$\text{either} \quad \liminf_{t\to\infty} \frac{\rho_i(t)}{r_i(t)} > 0 \quad \text{or} \quad \liminf_{t\to\infty} \frac{\sum_{\ell=1}^{n_0} \alpha_{ii\ell}(t)}{r_i(t)} > 0, \quad i = 1, \ldots, n \tag{5.3.7}$$

are satisfied. Therefore Theorem 5.2.4 has the following consequence.

Corollary 5.3.2. *Assume that that* $\tau_{ij\ell}$ *satisfies* $(\boldsymbol{A_0})$, r_i *and* α_{ij} *satisfy* $(\boldsymbol{A_1})$ *and* $(\boldsymbol{A_2})$, $q_i \in \mathbb{N}$, $\rho_i \in C(\mathbb{R}_+, \mathbb{R}_+)$ *satisfies* $\sup_{t>0} \frac{\rho_i(t)}{r_i(t)} < \infty$, $1 \le i \le n$, *(5.3.6) and (5.3.7) hold. Then, for any initial function* $\varphi = (\varphi_1, \varphi_2, ..., \varphi_n) \in C_+^n$ *, the solutions* $x(t) = x(\varphi)(t) = (x_1(\varphi)(t), ..., x_n(\varphi)(t))$ *of the IVP (5.3.4) and (5.3.5) satisfy*

$$\underline{x}_i^* \le \liminf_{t\to\infty} x_i(\phi) \le \limsup_{t\to\infty} x_i(\phi) \le \overline{x}_i^*, \qquad 1 \le i \le n, \tag{5.3.8}$$

where $(\underline{x}_1^*, ..., \underline{x}_n^*)$ *is the unique positive solution of the system*

$$x_i^{q_i} = \sum_{j=1}^{n} \underline{m}_{ij} x_j^{p_{ij}} + \underline{l}_i, \qquad 1 \le i \le n, \tag{5.3.9}$$

and $(\overline{x}_1^*, ..., \overline{x}_n^*)$ *is the unique positive solution of the system*

$$x_i^{q_i} = \sum_{j=1}^{n} \overline{m}_{ij} x_j^{p_{ij}} + \overline{l}_i, \qquad 1 \le i \le n, \tag{5.3.10}$$

respectively, where $\underline{m}_{ij}$, $\overline{m}_{ij}$, $\underline{l}_i$ *and* $\bar{l}_i$ *are defined in (5.2.23) and (5.2.24) for* $1 \le i, j \le n$.

We remark that the condition (5.3.6) in Corollary 5.3.2 can be weakened. Next we study the asymptotic equivalence of positive solutions for a special form of the System (5.3.4). We consider the IVP

$$\dot{x}_i(t) = \sum_{j=1}^{n} \sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t) x_j(t - \tau_{ij\ell}(t)) - r_i(t) x_i^{q_i}(t) + \rho_i(t), \qquad t \ge 0, \qquad 1 \le i \le n, \tag{5.3.11}$$

with the initial condition

$$x_i(t) = \varphi_i(t), \qquad -\tau \le t \le 0, \qquad 1 \le i \le n, \tag{5.3.12}$$

where $\tau > 0$, $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n) \in C_+^n$, $\alpha_{ij\ell}, \tau_{ij\ell}, r_i \in C(\mathbb{R}_+, \mathbb{R}_+)$, $1 \le i, j \le n$, $1 \le \ell \le n_0$ and $q_i \in \mathbb{N}$, $q_i > 1$, $1 \le i \le n$.

Remark. Equation (5.3.9) corresponding to (5.3.11) has the form

$$x_i^{q_i} = \sum_{j=1}^{n} \underline{m}_{ij} x_j + \underline{l}_i, \qquad 1 \le i \le n.$$

Therefore

$$x_i(x_i^{q_i - 1} - \underline{m}_{ii}) = \sum_{\substack{j=1 \\ j \ne i}}^{n} \underline{m}_{ij} x_j + \underline{l}_i, \qquad 1 \le i \le n.$$

So its positive solution $(\underline{x}_1^*, \dots, \underline{x}_n^*)$ satisfies $\underline{x}_i^* \ge \underline{m}_{ii}^{\frac{1}{q_i - 1}}$, hence Corollary 5.3.2 yields that for every $\varphi \in C_+^n$ the solution $x_i(\varphi)(t)$ of (5.3.11)-(5.3.12) satisfies

$$\liminf_{t \to \infty} x_i(\varphi)(t) \ge \underline{x}_i^* \ge \underline{m}_{ii}^{\frac{1}{q_i - 1}}, \qquad 1 \le i \le n. \tag{5.3.13}$$

Theorem 5.3.3. *Suppose that $\tau_{ij\ell}$, r_i and $\alpha_{ij\ell}$ satisfy ($\boldsymbol{A_0}$), ($\boldsymbol{A_1}$) and ($\boldsymbol{A_2}$), $\rho_i \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies $\sup\limits_{t>0} \frac{\rho_i(t)}{r_i(t)} < \infty$, $1 \le i \le n$, and*

$$\sum_{j=1}^{n} \overline{m}_{ij} < q_i \underline{m}_{ii}, \qquad q_i > 1, \quad 1 \le i \le n. \tag{5.3.14}$$

Then, for any initial functions $\varphi, \psi \in C_+^n$, the corresponding solutions $x(\varphi)(t)$ and $x(\psi)(t)$ of the IVP (5.3.11) and (5.3.12) satisfy

$$\lim_{t\to\infty} \Big(x_i(\varphi)(t) - x_i(\psi)(t) \Big) = 0, \qquad 1 \le i \le n, \tag{5.3.15}$$

i.e., any positive solutions of Eq. (5.3.11) are asymptotically equivalent.

Proof. See Appendix A. □

5.4 Applications to some population models

In this section, we give some applications to some population models which illustrate the applicability of our main results.

Next, we consider again the population model (5.1.9):

$$\begin{aligned} \dot{x}_i(t) \;\; &= \;\; \sum_{\ell=1}^{n_0} \frac{\lambda_{i\ell}(t) x_i(t - \tau_{i\ell}(t))}{1 + \gamma_{i\ell}(t) x_i(t - \tau_{i\ell}(t))} + \sum_{\substack{j=1 \\ j \ne i}}^{n} a_{ij}(t) x_j(t - \sigma_{ij}(t)) \\ &\quad - \mu_i(t) x_i(t) - \kappa_i(t) x_i^2(t), \qquad t \ge 0, \quad 1 \le i \le n, \end{aligned} \tag{5.4.1}$$

with the initial condition

$$x_i(t) = \varphi_i(t), \qquad -\tau \le t \le 0, \qquad 1 \le i \le n. \tag{5.4.2}$$

We assume that $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n) \in C_0^n$, where $C_0 := \{\psi \in C([-\tau, 0], \mathbb{R}_+) : \psi(t) > 0, \ -\tau \le t \le 0\}$. We note that $C_0 \subset C_+$.

The permanence of positive solutions of (5.4.1) was investigated in [32] for the case when the delays in the model can be unbounded. Next, we show that, for the bounded delay case, our Theorem 5.2.4 gives permanence of the positive solutions for this model under weak conditions. We note that we do not need the boundedness of the functions $\lambda_{i\ell}$, a_{ij}, μ_i and κ_i which was assumed in [32].

Corollary 5.4.1. *Assume that* $\lambda_{i\ell}, \gamma_{i\ell}, a_{ij}, \mu_i, \kappa_i \in C(\mathbb{R}_+, \mathbb{R}_+)$, *and* $\tau_{i\ell}, \sigma_{ij} \in C(\mathbb{R}_+, \mathbb{R}_+)$ *with* $0 \le \tau_{i\ell}(t) \le \tau$ *and* $0 \le \sigma_{ij}(t) \le \tau$ *for* $t \ge 0$, $1 \le i \ne j \le n$ *and* $\ell = 1, \dots, n_0$. *Moreover, we assume that there exist positive constants* $\underline{\gamma}_i, \overline{\gamma}_i, \underline{\pi}_i$ *and* $\overline{\pi}_i$ *such that, for all* $1 \le i \ne j \le n$ *and* $1 \le \ell \le n_0$,

$$0 < \underline{\gamma}_i \le \gamma_{i\ell}(t) \le \overline{\gamma}_i, \quad 0 < \underline{\pi}_i \le \frac{\kappa_i(t)}{\mu_i(t)} \le \overline{\pi}_i, \ t > 0 \quad \text{and} \quad \int_0^\infty \mu_i(t)\, dt = \infty, \tag{5.4.3}$$

and

$$\sup_{t>0} \frac{\sum_{\ell=1}^{n_0} \lambda_{i\ell}(t)}{\mu_i(t)} < \infty, \quad \sup_{t>0} \frac{a_{ij}(t)}{\mu_i(t)} < \infty, \ j \ne i, \quad \text{and} \quad \liminf_{t\to\infty} \frac{\sum_{\ell=1}^{n_0} \lambda_{i\ell}(t)}{\mu_i(t)} > 1. \tag{5.4.4}$$

Then, for any initial function $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n) \in C_0^n$, *the solution* $x(\varphi)(t) = (x_1(\varphi)(t), \dots, x_n(\varphi)(t))$ *of the IVP (5.4.1) and (5.4.2) satisfies*

$$\underline{x}_i^* \le \liminf_{t\to\infty} x_i(\varphi)(t) \le \limsup_{t\to\infty} x_i(\varphi)(t) \le \overline{x}_i^*, \qquad 1 \le i \le n, \tag{5.4.5}$$

where $(\underline{x}_1^*, \dots, \underline{x}_n^*)$ *is the unique positive solution of the algebraic system*

$$x_i + \overline{\pi}_i x_i^2 = \frac{\underline{m}_{ii} x_i}{1 + \overline{\gamma}_i x_i} + \sum_{\substack{j=1\\ j\ne i}}^{n} \underline{m}_{ij} x_j, \qquad 1 \le i \le n, \tag{5.4.6}$$

and $(\overline{x}_1^*, \dots, \overline{x}_n^*)$ *is the unique positive solution of the algebraic system*

$$x_i + \underline{\pi}_i x_i^2 = \frac{\overline{m}_{ii} x_i}{1 + \underline{\gamma}_i x_i} + \sum_{\substack{j=1\\ j\ne i}}^{n} \overline{m}_{ij} x_j, \qquad 1 \le i \le n, \tag{5.4.7}$$

respectively, where $\underline{m}_{ii} := \liminf_{t\to\infty} \frac{\sum_{\ell=1}^{n_0} \lambda_{i\ell}(t)}{\mu_i(t)}$, $\overline{m}_{ii} := \limsup_{t\to\infty} \frac{\sum_{\ell=1}^{n_0} \lambda_{i\ell}(t)}{\mu_i(t)}$, $1 \le i \le n$, *and* $\underline{m}_{ij} := \liminf_{t\to\infty} \frac{a_{ij}(t)}{\mu_i(t)}$, $\overline{m}_{ij} := \limsup_{t\to\infty} \frac{a_{ij}(t)}{\mu_i(t)}$ *for* $1 \le i \ne j \le n$.

Proof. All conditions of Lemma 5.2.1 hold for the System (5.4.1), therefore it implies that $x_i(t) = x_i(\varphi)(t) > 0$ for $t \ge 0$ and $i = 1, \dots, n$. Since we assumed that $\varphi_i \in C_0$ for all $i = 1, \dots, n$, it follows $x_i(t - \tau_{i\ell}(t)) > 0$ for $t \ge 0$ and $i = 1, \dots, n$. From (5.4.3), we have $\gamma_{i\ell}(t) \le \overline{\gamma}_i$ and $\frac{\kappa_i(t)}{\mu_i(t)} \le \overline{\pi}_i$, for $t > 0$. Thus, we get from (5.4.1) for $t \geq 0$ and $i = 1, \ldots, n$ that

$$\dot{x}_i(t) \geq \sum_{\ell=1}^{n_0} \frac{\lambda_{i\ell}(t)x_i(t-\tau_{i\ell}(t))}{1+\overline{\gamma}_i x_i(t-\tau_{i\ell}(t))} + \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}(t)x_j(t-\sigma_{ij}(t)) - \mu_i(t)[x_i(t) + \overline{\pi}_i x_i^2(t)].$$

By Theorem 2.1.2, we have $x_i(t) \geq y_i(t)$ for $t \geq 0$ and $i = 1, \ldots, n$, where $y_i(t)$ is the positive solution of the differential equation

$$\begin{aligned} \dot{y}_i(t) &= \sum_{\ell=1}^{n_0} \frac{\lambda_{i\ell}(t)y_i(t-\tau_{i\ell}(t))}{1+\overline{\gamma}_i y_i(t-\tau_{i\ell}(t))} + \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}(t)y_j(t-\sigma_{ij}(t)) \\ &\quad -\mu_i(t)[y_i(t) + \overline{\pi}_i y_i^2(t)], \qquad 1 \leq i \leq n, \end{aligned} \tag{5.4.8}$$

with the initial condition

$$y_i(t) = \varphi_i(t), \qquad -\tau \leq t \leq 0, \quad 1 \leq i \leq n. \tag{5.4.9}$$

Next, we check that conditions $(\mathbf{A_0})$–$(\mathbf{A_6})$ of Theorem 5.2.4 are satisfied for the System (5.4.8). First note that we can rewrite (5.4.8) in the form (5.2.1) with

$$\begin{aligned} \alpha_{ij\ell}(t) &:= \begin{cases} \lambda_{i\ell}(t), & j = i, \quad \ell = 1, \ldots, n_0, \\ a_{ij}(t), & j \neq i, \quad \ell = 1, \\ 0, & j \neq i, \quad \ell \neq 1, \end{cases} \\ h_{ij}(u) &:= \begin{cases} \frac{u}{1+\overline{\gamma}_i u}, & j = i, \\ u, & j \neq i, \end{cases} \\ \tau_{ij\ell}(t) &:= \begin{cases} \tau_{i\ell}(t), & j = i, \quad \ell = 1, \ldots, n_0, \\ \sigma_{ij}(t), & j \neq i, \quad \ell = 1, \\ 0, & j \neq i, \quad \ell \neq 1, \end{cases} \end{aligned}$$

and $r_i(t) := \mu_i(t)$, $f_i(u) := u + \overline{\pi}_i u^2$ and $\rho_i(t) := 0$, $1 \leq i, j \leq n$. We have $\lim_{u\to 0^+} \frac{f_i(u)}{h_{ii}(u)} = \lim_{u\to 0^+} \frac{(u+\overline{\pi}_i u^2)(1+\overline{\gamma}_i u)}{u} = 1$ and $\lim_{u\to\infty} \frac{h_{ij}(u)}{f_i(u)} = 0$ for all $1 \leq i, j \leq n$.

Therefore, by our assumptions (5.4.3) and (5.4.4), we can see that conditions $(\mathbf{A_0})$–$(\mathbf{A_5})$ hold. To check condition $(\mathbf{A_6})$, we observe that

$$\frac{f_i(u)}{h_{ij}(u)} = \begin{cases} (1+\overline{\pi}_i u)(1+\overline{\gamma}_i u), & j = i, \\ 1+\overline{\pi}_i u, & j \neq i, \end{cases}$$

is strictly increasing and

$$\frac{h_{jj}(u)}{h_{ij}(u)} = \frac{u}{u(1+\overline{\gamma}_i u)} = \frac{1}{1+\overline{\gamma}_i u}$$

is strictly decreasing on $(0,\infty)$, for each $1 \le i \ne j \le n$. We see that $\underline{m}_{jj} = \liminf_{t\to\infty} \frac{\sum_{\ell=1}^{n_0} \lambda_{j\ell}(t)}{\mu_j(t)} > 1$ by (5.4.4), and $\frac{h_{jj}(u)}{h_{ij}(u)}$ is strictly decreasing on $(0,\infty)$, for all $j \ne i$. Hence conditions $(\mathbf{A_6})$ (i), (ii) and (iii) are satisfied, and we can apply Theorem 5.2.4 (i) to the System (5.4.8). Therefore we get the lower estimates $\liminf_{t\to\infty} x_i(\varphi)(t) \ge \liminf_{t\to\infty} y_i(\varphi)(t) \ge \underline{x}_i^*$, $1 \le i \le n$, where $(\underline{x}_1^*, \dots, \underline{x}_n^*)$ is the unique positive solution of the algebraic system (5.4.6). Similarly, we can get the upper estimates $\limsup_{t\to\infty} x_i(\varphi)(t) \le \overline{x}_i^*$, $1 \le i \le n$, where $(\overline{x}_1^*, \dots, \overline{x}_n^*)$ is the unique positive solution of the algebraic system (5.4.7). □

Now, we consider a time-dependent version of the n-dimensional Nicholson's blowflies system (5.1.8) for $t \ge 0$:

$$\dot{x}_i(t) = \sum_{\ell=1}^{n_0} b_{i\ell}(t) x_i(t-\sigma_{i\ell}(t)) e^{-x_i(t-\sigma_{i\ell}(t))} + \sum_{\substack{j=1\\ j\ne i}}^{n} a_{ij}(t) x_j(t) - d_i(t) x_i(t),\ 1 \le i \le n \tag{5.4.10}$$

with the initial condition

$$x_i(t) = \varphi_i(t), \qquad -\tau \le t \le 0, \qquad 1 \le i \le n, \tag{5.4.11}$$

where $\tau > 0$, $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n) \in C_+^n$, $b_{i\ell}, a_{ij}, d_i \in C(\mathbb{R}_+, \mathbb{R}_+)$, and $\sigma_{i\ell} \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $0 \le \sigma_{i\ell}(t) \le \tau$ for $t \ge 0$, $1 \le i \ne j \le n$, $\ell = 1, \dots, n_0$. The persistence and permanence of the autonomous system (5.1.8) was investigated in [33]. Unfortunately, our method does not work for this population model, since the function ue^{-u} is not monotone increasing, and so condition $(\mathbf{A_4})$ of our main Theorem 5.2.4 is not satisfied for (5.4.10). But we can apply our method to get an upper bound of the limit superior of the solutions of (5.4.10). We formulate this result next.

Corollary 5.4.2. *Assume* $b_{i\ell}, a_{ij}, d_i \in C(\mathbb{R}_+, \mathbb{R}_+)$, *and* $\sigma_{i\ell} \in C(\mathbb{R}_+, \mathbb{R}_+)$ *with* $0 \le$

$\sigma_{i\ell}(t) \le \tau$ for $t \ge 0$, $1 \le i \ne j \le n$ and $\ell = 1, \dots, n_0$. Moreover, we assume that, for all $1 \le i, j \le n$,

$$d_i(t) > 0, \quad t > 0 \qquad \text{and} \qquad \int_0^\infty d_i(t)\,dt = \infty, \tag{5.4.12}$$

$$\sup_{t>0} \frac{\sum_{\ell=1}^{n_0} b_{i\ell}(t)}{d_i(t)} < \infty \qquad \text{and} \qquad \sup_{t>0} \frac{a_{ij}(t)}{d_i(t)} < \infty, \quad j \ne i, \tag{5.4.13}$$

and

$$\liminf_{t\to\infty} \frac{\sum_{\ell=1}^{n_0} b_{i\ell}(t)}{d_i(t)} > 1 \qquad \text{and} \qquad \sum_{\substack{j=1\\ j\ne i}}^{n} \limsup_{t\to\infty} \frac{a_{ij}(t)}{d_i(t)} < 1. \tag{5.4.14}$$

Then, for any initial function $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n) \in C_+^n$, the solution $x(\varphi)(t) = (x_1(\varphi)(t), \dots, x_n(\varphi)(t))$ of the IVP (5.4.10) and (5.4.11) satisfies

$$x_i(\varphi)(t) > 0, \quad t \ge 0, \quad \text{and} \quad \limsup_{t\to\infty} x_i(\varphi)(t) \le \overline{x}_i^*, \qquad 1 \le i \le n, \tag{5.4.15}$$

where $(\overline{x}_1^*, \dots, \overline{x}_n^*)$ is the unique positive solution of the algebraic system

$$x_i = \overline{m}_{ii} H(x_i) + \sum_{\substack{j=1\\ j\ne i}}^{n} \overline{m}_{ij} x_j, \qquad 1 \le i \le n, \tag{5.4.16}$$

where $\overline{m}_{ii} := \limsup_{t\to\infty} \frac{\sum_{\ell=1}^{n_0} b_{i\ell}(t)}{d_i(t)}$, $1 \le i \le n$, and $\overline{m}_{ij} := \limsup_{t\to\infty} \frac{a_{ij}(t)}{d_i(t)}$ for $1 \le i \ne j \le n$, and

$$H(u) := \begin{cases} ue^{-u}, & u \le 1, \\ \frac{1}{e}, & u > 1. \end{cases} \tag{5.4.17}$$

Proof. All conditions of Lemma 5.2.1 hold for the System (5.4.10), therefore it implies that $x_i(\varphi)(t) > 0$ for $t \ge 0$ and $i = 1, \dots, n$. We have $ue^{-u} \le H(u)$ for $u \ge 0$, therefore (5.4.10) yields

$$\dot{x}_i(t) \le \sum_{\ell=1}^{n_0} b_{i\ell}(t) H(x_i(t - \sigma_{i\ell}(t))) + \sum_{\substack{j=1\\ j\ne i}}^{n} a_{ij}(t) x_j(t) - d_i(t) x_i(t), \qquad 1 \le i \le n.$$

By Theorem 2.1.2, we have $x_i(t) \le y_i(t)$ for $t \ge 0$, $i = 1, \dots, n$, where $y_i(t)$ is the positive solution of the differential equation

$$\dot{y}_i(t) = \sum_{\ell=1}^{n_0} b_{i\ell}(t) H(y_i(t-\sigma_{i\ell}(t))) + \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}(t) y_j(t) - d_i(t) y_i(t), \qquad 1 \le i \le n, \tag{5.4.18}$$

with the initial condition

$$y_i(t) = \varphi_i(t), \qquad -\tau \le t \le 0, \quad 1 \le i \le n. \tag{5.4.19}$$

Next, we check that $(\mathbf{A_0})$–$(\mathbf{A_6})$ of Theorem 5.2.4 are satisfied for the System (5.4.18). First note that we can rewrite (5.4.18) in the form (5.2.1) with

$$\alpha_{ij\ell}(t) := \begin{cases} b_{i\ell}(t), & j = i, \quad \ell = 1, \dots, n_0, \\ a_{ij}(t), & j \neq i, \quad \ell = 1, \\ 0, & j \neq i, \quad \ell \neq 1, \end{cases}$$

$$h_{ij}(u) := \begin{cases} H(u), & j = i, \\ u, & j \neq i, \end{cases}$$

$$\tau_{ij\ell}(t) := \begin{cases} \sigma_{i\ell}(t), & j = i, \quad \ell = 1, \dots, n_0, \\ 0, & \text{otherwise}, \end{cases}$$

and $r_i(t) := d_i(t)$, $f_i(u) := u$ and $\rho_i(t) := 0$, $1 \le i, j \le n$. We have

$$\lim_{u\to 0^+} \frac{f_i(u)}{h_{ii}(u)} = \lim_{u\to 0^+} \frac{u}{H(u)} = 1 \quad \text{and} \quad \lim_{u\to\infty} \frac{h_{ij}(u)}{f_i(u)} = \begin{cases} 0, & j = i, \\ 1, & j \neq i \end{cases}$$

for $1 \le i, j \le n$. Thus, by our assumptions (5.4.12), (5.4.13) and (5.4.14), we can see that conditions $(\mathbf{A_0})$–$(\mathbf{A_5})$ hold. To check condition $(\mathbf{A_6})$, we observe that

$$\frac{f_i(u)}{h_{ij}(u)} = \begin{cases} e^u, & u \le 1, \ j = i, \\ eu, & u > 1, \ j = i, \\ 1, & u > 1, \ j \neq i, \end{cases}$$

is increasing and

$$\frac{h_{jj}(u)}{h_{ij}(u)} = \frac{H(u)}{h_{ij}(u)} = \begin{cases} e^{-u}, & u \le 1, \ j \neq i, \\ \frac{1}{eu}, & u > 1, \ j \neq i, \end{cases}$$

is strictly decreasing on $(0, \infty)$, for each $1 \le i, j \le n$. Moreover, for each $1 \le i \le n$,

$\frac{f_i(u)}{h_{ii}(u)}$ is strictly increasing on $(0,\infty)$. For each $j=1,\dots,n$, $\overline{m}_{jj} \geq \liminf\limits_{t\to\infty} \frac{\sum_{\ell=1}^{n_0} b_{j\ell}(t)}{d_j(t)} > 1$ by (5.4.14), and $\frac{h_{jj}(u)}{h_{ij}(u)}$ is strictly decreasing on $(0,\infty)$, for all $j \neq i$. Hence conditions $(\mathbf{A_6})$ (i), (iv) and (v) are satisfied, and we can apply Theorem 5.2.4 (ii) to the System (5.4.18). Therefore we can obtain the upper estimates $\limsup\limits_{t\to\infty} x_i(\varphi)(t) \leq \limsup\limits_{t\to\infty} y_i(\varphi)(t) \leq \overline{x}_i^*$, $1 \leq i \leq n$, where $(\overline{x}_1^*,\dots,\overline{x}_n^*)$ is the unique positive solution of the algebraic system (5.4.16). □

5.5 Examples

In this section, we give some examples with numerical simulations to illustrate our main results.

Example 5.5.1. Consider the following system of nonlinear differential equations in the three dimensions, for $t \geq 0$,

$$\begin{aligned}
\dot{x}_1(t) &= t^{0.1}(1+\cos t)x_1(t-2) + t^{0.1}x_1(t-1.5) + t^{0.1}x_2^2(t-0.05) \\
&\quad + t^{0.1}x_2^2(t-3) + t^{0.1}(2+2\sin t)x_3^3(t-0.5) \\
&\quad + t^{0.1}x_3^3(t-2.4) + t^{0.1}x_3^3(t-2.5) - 2t^{0.1}x_1^4(t) \\
&\quad + 0.2t^{0.1}(1.2+\sin t), \\
\dot{x}_2(t) &= x_1(t-1.5) + 2x_1(t-0.5) + x_1(t-0.4) \\
&\quad + 6(10+\cos t)x_2(t-0.05) + (3+3\cos t)x_3^2(t-0.09) \\
&\quad + 2x_3^2(t-1.3) - x_2^3(t) + 4.5 + \cos t, \\
\dot{x}_3(t) &= 5x_1^2(t-1.9) + 2x_1^3(t-0.2) + x_1^3(t-0.3) + 10x_2(t-1.2) \\
&\quad + (2+5\sin t)x_2(t-5) + 6x_3^2(t-0.01) + 4x_3^2(t-1) \\
&\quad - 2x_3^3(t) + 4.5 + 2\cos t.
\end{aligned} \tag{5.5.1}$$

Note that the conditions of Corollary 5.3.2 are satisfied for (5.5.1). So, we see from Corollary 5.3.2 that

$$\liminf_{t\to\infty} x_1(t) \geq \underline{x}_1^*, \quad \liminf_{t\to\infty} x_2(t) \geq \underline{x}_2^* \quad \text{and} \quad \liminf_{t\to\infty} x_1(t) \geq \underline{x}_3^*,$$

where $(\underline{x}_1^*, \underline{x}_2^*, \underline{x}_3^*)$ is the unique positive solution of the algebraic system

$$\begin{aligned} x_1^4 &= 0.5x_1 + x_2^2 + x_3^3 + 0.02, \\ x_2^3 &= 4x_1 + 54x_2 + 2x_3^2 + 3.5, \\ x_3^3 &= 4x_1^2 + 3.5x_2 + 5x_3^2 + 1.25. \end{aligned} \tag{5.5.2}$$

We solve the System (5.5.2) numerically by the fixed point iteration

$$\begin{aligned} \underline{x}_1^{(k+1)} &= \sqrt[4]{0.5\underline{x}_1^{(k)} + (\underline{x}_2^{(k)})^2 + (\underline{x}_3^{(k)})^3 + 0.02}, \\ \underline{x}_2^{(k+1)} &= \sqrt[3]{4\underline{x}_1^{(k)} + 54\underline{x}_2^{(k)} + 2(\underline{x}_3^{(k)})^2 + 3.5}, \\ \underline{x}_3^{(k+1)} &= \sqrt[3]{4(\underline{x}_1^{(k)})^2 + 3.5\underline{x}_2^{(k)} + 5(\underline{x}_3^{(k)})^2 + 1.25}. \end{aligned} \tag{5.5.3}$$

We compute the sequence defined by the iteration (5.5.3) starting from the initial value $(\underline{x}_1^{(0)}, \underline{x}_2^{(0)}, \underline{x}_3^{(0)}) = (0, 0, 0)$. The first ten terms of this sequence are displayed in Table 5.5.1. We can observe that the sequence is convergent, and its limit is $(\underline{x}_1^*, \underline{x}_2^*, \underline{x}_3^*) = (4.5960\ldots, 8.3147\ldots, 7.2095\ldots)$.

Similarly, we can see that

$$\limsup_{t\to\infty} x_1(t) \le \overline{x}_1^*, \quad \limsup_{t\to\infty} x_2(t) \le \overline{x}_2^* \quad \text{and} \quad \limsup_{t\to\infty} x_1(t) \le \overline{x}_3^*,$$

where $(\overline{x}_1^*, \overline{x}_2^*, \overline{x}_3^*)$ is the unique positive solution of the algebraic system

$$\begin{aligned} x_1^4 &= 1.5x_1 + x_2^2 + 3x_3^3 + 0.22, \\ x_2^3 &= 4x_1 + 66x_2 + 8x_3^2 + 5.5, \\ x_3^3 &= 4x_1^2 + 8.5x_2 + 5x_3^2 + 3.25. \end{aligned} \tag{5.5.4}$$

We solve the System (5.5.4) numerically by a fixed point iteration defined similarly to (5.5.3) from the starting value $(0, 0, 0)$. The numerical results can be seen in Table 5.5.2. We conclude that $(\overline{x}_1^*, \overline{x}_2^*, \overline{x}_3^*) = (6.7840\ldots, 11.1161\ldots, 8.7126\ldots)$. Therefore Corollary 5.3.2 yields

$$\begin{aligned} 4.5960\ldots \le \liminf_{t\to\infty} x_1(t) \le \limsup_{t\to\infty} x_1(t) \le 6.7840\ldots, \\ 8.3147\ldots \le \liminf_{t\to\infty} x_2(t) \le \limsup_{t\to\infty} x_2(t) \le 11.1161\ldots, \\ 7.2095\ldots \le \liminf_{t\to\infty} x_3(t) \le \limsup_{t\to\infty} x_3(t) \le 8.7126\ldots. \end{aligned} \tag{5.5.5}$$

We plotted the numerical solution of the System (5.5.1) in Figure 5.5.1 corresponding to the constant initial functions $(\varphi_1(t), \varphi_2(t), \varphi_3(t)) = (2.5, 6, 2.5)$ and

$(\varphi_1(t), \varphi_2(t), \varphi_3(t)) = (3.5, 8, 4)$. The horizontal lines in Figure 5.5.1 correspond to the upper and lower bounds listed in (5.5.5), respectively. We also observe that the difference of the components of the two solutions converges to zero, i.e., the two solutions are asymptotically equivalent. The numerical results demonstrate the theoretical bounds (5.5.5). $\square$

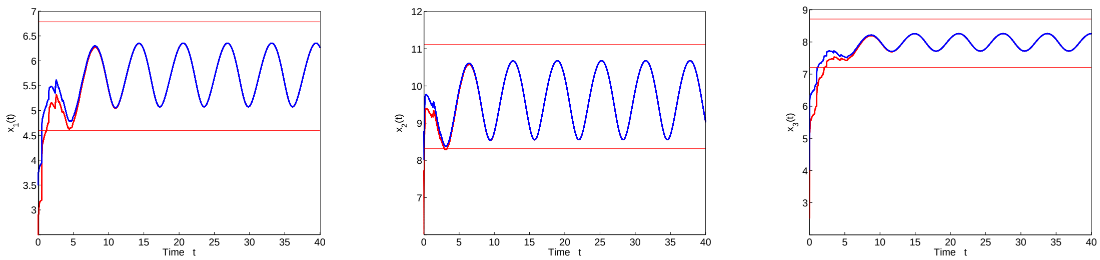

Figure 5.5.1: Numerical solution of the System (5.5.1).

k	$\underline{x}_1^{(k)}$	$\underline{x}_2^{(k)}$	$\underline{x}_3^{(k)}$
0	0	0	0
1	0.3761	1.7105	1.9834
2	1.8185	4.8060	3.7077
3	3.6353	7.5553	5.9214
4	4.0406	7.9252	6.4602
5	4.4130	8.1962	6.9628
6	4.5364	8.2765	7.1294
7	4.5767	8.3023	7.1836
8	4.5958	8.3146	7.2092
9	4.5960	8.3147	7.2095
10	4.5960	8.3147	7.2095

Table 5.5.1: Numerical solution of the System (5.5.2)

k	$\overline{x}_1^{(k)}$	$\overline{x}_2^{(k)}$	$\overline{x}_3^{(k)}$
0	0	0	0
1	0.6849	2.0198	2.8145
2	2.9151	5.9799	5.0354
3	5.5288	9.7858	7.5194
4	6.4086	10.7362	8.3557
5	6.6740	11.0053	8.6081
6	6.7520	11.0838	8.6822
7	6.7747	11.1067	8.7038
8	6.7839	11.1159	8.7125
9	6.7840	11.1161	8.7126
10	6.7840	11.1161	8.7126

Table 5.5.2: Numerical solution of the System (5.5.4)

Example 5.5.2. Consider the following system of nonlinear differential equations in the two dimensions, for $t \geq 0$,

$$\begin{aligned} \dot{x}_1(t) &= (1.7 + 0.2\cos t)x_1(t-2) + (0.25 + 0.1\sin t)x_2(t-1.5) \\ &\quad -0.5x_1^2(t) + 8 + 2\cos t, \\ \dot{x}_2(t) &= (0.02 + 0.01\sin t)x_1(t-0.3) + (1.2 + 0.2\cos t)x_2(t-10) \\ &\quad -0.2x_2^2(t) + 2.2 + 2\sin t. \end{aligned} \tag{5.5.6}$$

Note that the conditions of Theorem 5.3.3 are satisfied for (5.5.6), where $\underline{m}_{11} = 3$, $\overline{m}_{11} = 3.8$, $\overline{m}_{12} = 0.7$, $\underline{m}_{22} = 5$, $\overline{m}_{21} = 0.15$ and $\overline{m}_{22} = 7$ satisfy (5.3.14) for $i, j = 1, 2$. Also, using Corollary 5.3.2, we see that

$$\liminf_{t\to\infty} x_1(t) \geq \underline{x}_1^*, \quad \text{and} \quad \liminf_{t\to\infty} x_2(t) \geq \underline{x}_2^*,$$

where $(\underline{x}_1^*, \underline{x}_2^*)$ is the unique positive solution of the system

$$\begin{aligned} x_1^2 &= 3x_1 + 0.3x_2 + 12, \\ x_2^2 &= 0.05x_1 + 5x_2 + 1. \end{aligned} \tag{5.5.7}$$

We solve the System (5.5.7) numerically by a fixed point iteration

$$\begin{aligned} \underline{x}_1^{(k+1)} &= \sqrt{3\underline{x}_1^{(k)} + 0.3\underline{x}_2^{(k)} + 12}, \\ \underline{x}_2^{(k+1)} &= \sqrt{0.05\underline{x}_1^{(k)} + 5\underline{x}_2^{(k)} + 1}. \end{aligned} \tag{5.5.8}$$

We compute the sequence defined by the iteration (5.5.8) starting from the initial value $(0, 0)$. The first ten terms of this sequence are displayed in Table 5.5.3. We can observe that the sequence is convergent and its limit is $(\underline{x}_1^*, \underline{x}_2^*) = (5.4778\ldots, 5.2430\ldots)$.

Similarly, we can see that

$$\limsup_{t\to\infty} x_1(t) \leq \overline{x}_1^*, \quad \text{and} \quad \limsup_{t\to\infty} x_2(t) \leq \overline{x}_2^*,$$

where $(\overline{x}_1^*, \overline{x}_2^*)$ is the unique positive solution of the system

$$\begin{aligned} x_1^2 &= 3.8x_1 + 0.7x_2 + 20, \\ x_2^2 &= 0.15x_1 + 7x_2 + 21. \end{aligned} \tag{5.5.9}$$

We solve the System (5.5.9) numerically by a fixed point iteration defined similarly to (5.5.8) from the starting value $(0, 0)$. The numerical results can be seen in Table 5.5.4. We conclude that $(\overline{x}_1^*, \overline{x}_2^*) = (7.3921\ldots, 9.3616\ldots)$. Therefore Corollary 5.3.2

yields

$$
\begin{aligned}
5.4778\ldots \le \liminf_{t\to\infty} x_1(t) \le \limsup_{t\to\infty} x_1(t) \le 7.3921\ldots,\\
5.2430\ldots \le \liminf_{t\to\infty} x_2(t) \le \limsup_{t\to\infty} x_2(t) \le 9.3616\ldots.
\end{aligned} \tag{5.5.10}
$$

We plotted the numerical solution of the System (5.5.6) in Figure 5.5.2 corresponding to the initial functions $(\varphi_1(t), \varphi_2(t)) = (3, 2)$, $(\varphi_1(t), \varphi_2(t)) = (7, 7)$ and $(\varphi_1(t), \varphi_2(t)) = (9, 10)$. The horizontal lines in Figure 5.5.2 correspond to the upper and lower bounds listed in (5.5.10), respectively. We also observe that the difference of the components of every two solutions converges to zero, i.e., the two solutions are asymptotically equivalent which coincide (5.3.15) in Theorem 5.3.3. □

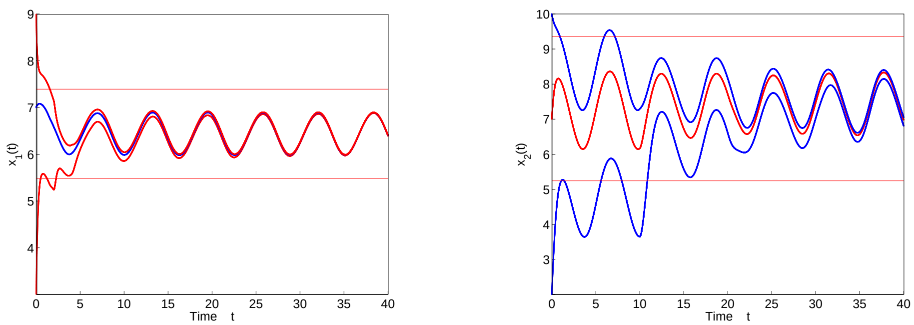

Figure 5.5.2: Numerical solution of the System (5.5.6).

Example 5.5.3. Consider the 2-dimensional population model:

$$
\begin{aligned}
\dot{x}_1(t) &= \tfrac{(1+0.8\cos t)x_1(t-2.05)}{1+(2+\sin(0.1t))x_1(t-2.05)} + \tfrac{2(1+0.5\cos t)x_1(t-1.5)}{1+(2+\sin(0.1t))x_1(t-1.5)} + 4x_2(t-1.8)\\
&\quad -3x_1(t) - (4+\sin t)x_1^2(t);\\
\dot{x}_2(t) &= \tfrac{2x_2(t-0.3)}{1+e^{\sin t}x_2(t-0.3)} + \tfrac{4x_2(t-1)}{1+e^{\sin t}x_2(t-1)} + (1+e^{\sin t})x_1(t-2.5)\\
&\quad -2x_2(t) - 2e^{2\sin t}x_2^2(t).
\end{aligned} \tag{5.5.11}
$$

Using Corollary 5.4.1, we see that

$$
\liminf_{t\to\infty} x_1(t) \ge \underline{x}_1^*, \quad \text{and} \quad \liminf_{t\to\infty} x_2(t) \ge \underline{x}_2^*,
$$

k	$\underline{x}_1^{(k)}$	$\underline{x}_2^{(k)}$
0	0	0
1	3.4641	1.0831
2	4.7663	2.5795
3	5.2031	3.7627
4	5.4246	4.8659
5	5.4659	5.1549
6	5.4721	5.2008
7	5.4751	5.2419
8	5.4777	5.2429
9	5.4778	5.2430
10	5.4778	5.2430

Table 5.5.3: Numerical solution of the System (5.5.7)

k	$\overline{x}_1^{(k)}$	$\overline{x}_2^{(k)}$
0	0	0
1	4.4721	4.6552
2	6.3445	7.3850
3	7.0199	8.5877
4	7.2586	9.0666
5	7.3436	9.2503
6	7.3744	9.3198
7	7.3918	9.3608
8	7.3920	9.3615
9	7.3921	9.3616
10	7.3921	9.3616

Table 5.5.4: Numerical solution of the System (5.5.9)

where $(\underline{x}_1^*, \underline{x}_2^*)$ is the unique positive solution of the system

$$\begin{aligned} x_1 + 1.66667x_1^2 &= \tfrac{0.4x_1}{1+3x_1} + 1.33333x_2, \\ x_2 + 7.3891x_2^2 &= \tfrac{3x_2}{1+2.7183x_2} + 0.68395x_1. \end{aligned} \tag{5.5.12}$$

We solve the System (5.5.12) numerically by a fixed point iteration

$$\begin{aligned} \underline{x}_1^{(k+1)} &= \sqrt{\tfrac{1}{1.66667}\left[\tfrac{0.4\underline{x}_1^{(k)}}{1+3\underline{x}_1^{(k)}} + 1.33333\underline{x}_2^{(k)} - \underline{x}_1^{(k)}\right]}, \\ \underline{x}_2^{(k+1)} &= \sqrt{\tfrac{1}{7.3891}\left[\tfrac{3\underline{x}_2^{(k)}}{1+2.7183\underline{x}_2^{(k)}} + 0.68395\underline{x}_1^{(k)} - \underline{x}_2^{(k)}\right]}. \end{aligned} \tag{5.5.13}$$

We compute the sequence defined by the iteration (5.5.13) starting from the initial value $(0, 0.1)$. The first ten terms of this sequence are displayed in Table 5.5.5. We can observe that the sequence is convergent and its limit is $(\underline{x}_1^*, \underline{x}_2^*) = (0.2493\ldots, 0.2219\ldots)$.

Similarly, we can see that

$$\limsup_{t\to\infty} x_1(t) \le \overline{x}_1^*, \quad \text{and} \quad \limsup_{t\to\infty} x_2(t) \le \overline{x}_2^*,$$

where $(\overline{x}_1^*, \overline{x}_2^*)$ is the unique positive solution of the system

$$\begin{aligned} x_1 + x_1^2 &= \tfrac{1.6x_1}{1+x_1} + 1.33333x_2, \\ x_2 + 0.1353x_2^2 &= \tfrac{3x_2}{1+0.3679x_2} + 1.85915x_1. \end{aligned} \tag{5.5.14}$$

We solve the System (5.5.14) numerically by a fixed point iteration defined similarly to (5.5.13) from the starting value $(0, 0.1)$. The numerical results can be seen in Table 5.5.6. We conclude that $(\overline{x}_1^*, \overline{x}_2^*) = (2.5077\ldots, 5.7392\ldots)$. Therefore Corollary 5.4.1 yields

$$\begin{aligned} 0.2493\ldots \le \liminf_{t\to\infty} x_1(t) \le \limsup_{t\to\infty} x_1(t) \le 2.5077\ldots, \\ 0.2219\ldots \le \liminf_{t\to\infty} x_2(t) \le \limsup_{t\to\infty} x_2(t) \le 5.7392\ldots. \end{aligned} \tag{5.5.15}$$

We plotted the numerical solution of the System (5.5.11) in Figure 5.5.3 corresponding to the initial functions $(\varphi_1(t), \varphi_2(t)) = (0.1, 0.02)$ and $(\varphi_1(t), \varphi_2(t)) = (3, 6)$. $\square$

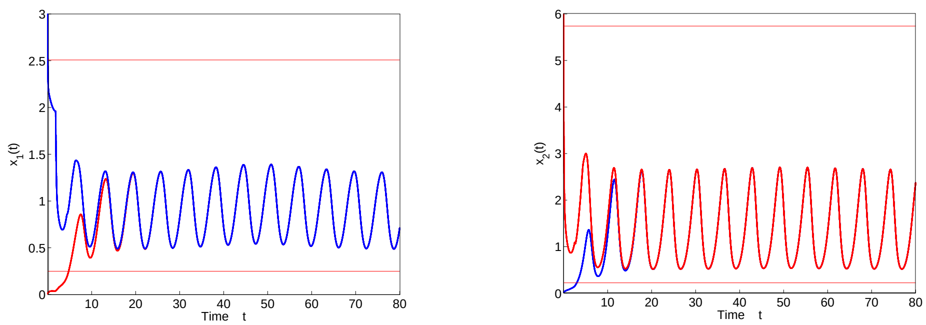

Figure 5.5.3: Numerical solution of the System (5.5.11).

k	$\underline{x}_1^{(k)}$	$\underline{x}_2^{(k)}$
0	0	0.1
1	0.2828	0.2111
2	0.2864	0.2287
3	0.2699	0.2258
4	0.2605	0.2241
5	0.2554	0.2231
6	0.2469	0.2214
7	0.2491	0.2218
8	0.2493	0.2219
9	0.2493	0.2219
10	0.2493	0.2219

Table 5.5.5: Numerical solution of the System (5.5.12)

k	$\overline{x}_1^{(k)}$	$\overline{x}_2^{(k)}$
0	0	0.1
1	0.3651	2.5332
2	2.6771	5.8300
3	2.4960	5.7462
4	2.5073	5.7382
5	2.5074	5.7388
6	2.5075	5.7389
7	2.5076	5.7390
8	2.5077	5.7391
9	2.5077	5.7392
10	2.5077	5.7392

Table 5.5.6: Numerical solution of the System (5.5.14)

Example 5.5.4. Consider the 2-dimensional Nicholson's population model:

$$
\begin{aligned}
\dot{x}_1(t) &= (1+0.8\cos t)x_1(t-2.05)e^{-x_1(t-2.05)} \\
&\quad +(4+\cos t)x_1(t-1.5)e^{-x_1(t-1.5)} + 0.3x_2(t) - 3x_1(t); \\
\dot{x}_2(t) &= 2x_2(t-0.3)e^{-x_2(t-0.3)} + 4x_2(t-1)e^{-x_2(t-1)} \\
&\quad +(1+0.2\sin t)x_1(t) - 2x_2(t).
\end{aligned}
\tag{5.5.16}
$$

Using Corollary 5.4.2, we can see that

$$\limsup_{t\to\infty} x_1(t) \le \overline{x}_1^*, \quad \text{and} \quad \limsup_{t\to\infty} x_2(t) \le \overline{x}_2^*,$$

where $(\overline{x}_1^*, \overline{x}_2^*)$ is the unique positive solution of the system

$$
\begin{aligned}
x_1 &= 2.2667H(x_1) + 0.1x_2, \\
x_2 &= 3H(x_2) + 0.6x_1,
\end{aligned}
\tag{5.5.17}
$$

where $H(u)$ is defined by (5.4.17). We solve the System (5.5.17) numerically by a fixed point iteration

$$
\begin{aligned}
\overline{x}_1^{(k+1)} &= 2.2667H(\overline{x}_1^{(k)}) + 0.1\overline{x}_2^{(k)}, \\
\overline{x}_2^{(k+1)} &= 3H(\overline{x}_2^{(k)}) + 0.6\overline{x}_1^{(k)}.
\end{aligned}
\tag{5.5.18}
$$

We compute the sequence defined by the iteration (5.5.18) starting from the initial value $(0, 0.1)$. The numerical results can be seen in Table 5.5.7. We conclude that $(\overline{x}_1^*, \overline{x}_2^*) = (1.0045\ldots, 1.7063\ldots)$. Therefore Corollary 5.4.2 yields

$$
\begin{aligned}
\liminf_{t\to\infty} x_1(t) \le \limsup_{t\to\infty} x_1(t) \le 1.0045\ldots, \\
\liminf_{t\to\infty} x_2(t) \le \limsup_{t\to\infty} x_2(t) \le 1.7063\ldots.
\end{aligned}
\tag{5.5.19}
$$

We plotted the numerical solution of the System (5.5.16) in Figure 5.5.4 corresponding to the initial functions $(\varphi_1(t), \varphi_2(t)) = (0.1, 0.8)$ and $(\varphi_1(t), \varphi_2(t)) = (1.5, 2)$. $\square$

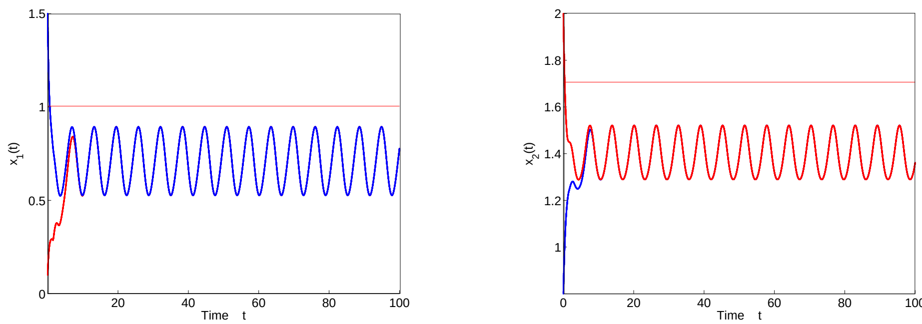

Figure 5.5.4: Numerical solution of the System (5.5.16).

k	$\overline{x}_1^{(k)}$	$\overline{x}_2^{(k)}$
0	0	0.1
1	0.0100	0.2775
2	0.1743	1.1283
3	0.4447	1.3704
4	0.7832	1.5735
5	0.9685	1.6848
6	1.0019	1.7048
7	1.0044	1.7062
8	1.0045	1.7063
9	1.0045	1.7063
10	1.0045	1.7063

Table 5.5.7: Numerical solution of the System (5.5.17)

Chapter 6

Conclusion

In this chapter we summarize the new results of the Thesis. Also we give the list of our publications and conference lectures related to this work.

6.1 New scientific results

Publications and conference abstracts are listed below. Some parts of this Thesis are published in (P1), (P2) and (P3).

Thesis 1: Sufficient conditions are given to guarantee the persistence and the uniform permanence of the positive solutions of nonlinear delay differential equations (related publication: (P1) and (P3)):

1.1: We establish sufficient conditions for the persistence of the positive solutions of the nonlinear scalar delay differential equation

$$\dot{x}(t) = r(t)\Big(g(t, x_t) - h(x(t))\Big), \qquad t \geq 0. \tag{6.1.1}$$

(Lemma 3.2.3)

1.2: We establish sufficient conditions to guarantee the uniform permanence of the positive solutions of the scalar Equation (6.1.1). (Theorem 3.2.4)

1.3: In several special cases of the scalar Equation (6.1.1) explicit upper and lower estimates of the limit super and limit inferior of the solutions are obtained. (Corollaries 3.3.1, 3.3.3, 3.3.4, 3.3.6, 3.3.7, 3.3.8, 3.3.9, 3.3.10)

1.4: Sufficient conditions are formulated for that all positive solutions of the scalar Equation (6.1.1) converge to a constant limit. (Corollary 3.2.5 and Corollary 3.3.5 for a special case)

1.5: We establish sufficient conditions to the uniform permanence of the positive solutions of a system of first order nonlinear delay differential equations

$$\dot{x}_i(t) = \sum_{j=1}^{n} \sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t) h_{ij}(x_j(t-\tau_{ij\ell}(t))) - r_i(t) f_i(x_i(t)) + \rho_i(t), \quad 1 \leq i \leq n. \tag{6.1.2}$$

(Theorem 5.2.4)

1.6: In several special cases of the System (6.1.2) (including n-dimensional population models with patch structure) upper and lower estimates of the limit super and limit inferior of the components of the solutions are obtained using the unique positive solutions of an associated system of nonlinear algebraic equations. (Corollaries 5.3.2, 5.4.1, 5.4.2)

1.7: Sufficient conditions are formulated for that all positive solutions of the System (6.1.2) converge to a constant limit. (Corollary 5.3.1)

Thesis 2: Sufficient conditions are given for the asymptotic equivalence of positive solutions of nonlinear delay differential equations (related publications (P1) and (P3)):

2.1: We establish sufficient conditions implying that for all $0 < p < q$, $q \geq 1$

all positive solutions of the equation

$$\dot{x}(t) = \sum_{k=1}^{n} \alpha_k(t) x^p(t - \tau_k(t)) - \beta(t) x^q(t)$$

are asymptotically equivalent. (Corollary 3.3.2)

2.2: We establish sufficient conditions to guarantee that all positive solutions of the system

$$\dot{x}_i(t) = \sum_{j=1}^{n} \sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t) x_j(t - \tau_{ij\ell}(t)) - r_i(t) x_i^{q_i}(t) + \rho_i(t), \quad q_i > 1, \quad 1 \leq i \leq n$$

are asymptotically equivalent. (Theorem 5.3.3)

Thesis 3: Sufficient conditions are given implying the existence and uniqueness of positive solutions of a system of nonlinear algebraic equations.(related publication: (P2)):

3.1: We establish sufficient conditions for the existence and uniqueness of the positive solutions of the nonlinear system of algebraic equations:

$$\gamma_i(x_i) = \sum_{j=1}^{n} g_{ij}(x_j), \qquad 1 \leq i \leq n. \tag{6.1.3}$$

(Theorem 4.2.1)

3.2: In several special cases of the System (6.1.3) we establish sufficient conditions for the unique positive solutions. (Corollaries 4.3.1, 4.3.2, 4.3.3, 4.3.4)

6.2 Publications and conference lectures

Publication and conference lectures of Nahed A. Mohamady are listed below. Some parts of this Thesis are published in (P1), (P2) and (P3).

6.2.1 Publications in refereed SCI journal (related to this Thesis)

(P1) István Győri, Ferenc Hartung, Nahed A. Mohamady, *On a Nonlinear Delay Population Model*, Applied Mathematics and Computation 270(2015)909-925. (**IF: 1.345**)

(P2) István Győri, Ferenc Hartung, Nahed A. Mohamady, *Existence and Uniqueness of Positive Solutions of a System of Nonlinear Algebraic Equations*, Period. Math. Hung., DOI 10.1007/s10998-016-0179-3, 2016. (**IF: 0.286**)

(P3) István Győri, Ferenc Hartung, Nahed A. Mohamady, *Boundedness of Positive Solutions of a System of Nonlinear Delay Differential Equations*, to appear in Discrete and Continuous Dynamical Systems- Series B. (**IF: 1.227**)

6.2.2 Publication in refereed journal (not related to this Thesis)

(P4) M. M. A. El-Sheikh, R. Sallam, N. Mohamady, *Oscillation Criteria for Second Order Nonlinear Neutral Differential Equations*, Electronic Journal of Differential Equations an Control Processes, ISSN 1817-2172, No. 3 (2011) 1-17.

(P5) M. M. A. EL-Sheikh, R. Sallam, N. Mohamady, *New Oscillation Criteria for General Neutral Delay Third Order Differential Equations*, International Journal of Mathematics and Computer Applications Research (IJMCAR) ISSN 2249-6955 Vol. 3, Issue 2, (Jun 2013) 183-190.

(P6) M. M. A. El-Sheikh, R. Sallam, N. Mohamady, *On the Oscillation of Third Order Neutral Delay Differential Equations*, Appl. Math. Inf. Sci. Lett. 1, No. 3,(2013)77-80.

(P7) M. M. A. EL-Sheikh, R. Sallam, Nahed A. Mohamady, *New Criteria for Oscillation of Second Order Nonlinear Dynamic Equations with Damping Time Scales*, International Journal of Research in Applied, Natural and Social Sciences (IJRANSS) ISSN(E): 2321-8851; ISSN(P): 2347-4580 Vol. 3, Issue 3 (Mar 2015) 79-86.

6.2.3 International conference presentations related to the Thesis

(C1) István Győri, Ferenc Hartung, **Nahed A. Mohamady**, *Boundedness of solutions of nonlinear delay differential equations*, 10th Colloquium on the Qualitative Theory of Differential Equations 2015, Bolyai Institute, University of Szeged, Hungary, July 1-4, 2015.

(C2) István Győri, Ferenc Hartung, **Nahed A. Mohamady**, *Persistence and Permanence of Nonlinear Delay Population Models*, The Second International Conference on New Horizons in Basic and Applied Science, Hurghada , Egypt, August 1-6, 2015.

(C3) István Győri, Ferenc Hartung, **Nahed A. Mohamady**, *Boundedness of positive solutions of a system of nonlinear delay differential equations*, O.D. EQUATIONS BRNO 2016, Faculty of Science, Masaryk University, Brno, Czech Republic, June 6 - 8, 2016.

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Appendix A

In this Appendix, we give some proofs of some of our results.

A.1 Proofs of some results in Chapter 3

Proof of Lemma 3.2.1. It is clear from condition ($\mathbf{H_2}$) that the IVP (3.2.3) and (3.2.4) has at least one solution for all $(T, y^*, c) \in (\mathbb{R}_+ \times (0, \infty) \times \mathbb{R}_+)$. Any of the corresponding solution $y(t) = y(T, y^*, c)(t)$ is considered. First we show that if $c \geq 0$ and $y^* \neq h^{-1}(c)$, then $y(t) \neq h^{-1}(c)$ for all $t \geq T$. Suppose that there exists a $t_1 > T$ such that $y(t_1) = h^{-1}(c)$. Thus, by separating variables in (3.2.3) and integrating from T to t_1, we get

$$\int_T^{t_1} \frac{\dot{y}(t)}{c - h(y(t))} dt = \int_T^{t_1} r(t) dt.$$

Introducing the new variable $u = y(t)$ and using ($\mathbf{H_2}$) with $v = h^{-1}(c)$ we get

$$\infty = \int_{y^*}^{h^{-1}(c)} \frac{1}{c - h(u)} du = \int_T^{t_1} r(t) dt,$$

which contradicts the continuity of r. Thus $y(t) \neq h^{-1}(c)$ for $t \geq T$. Note that for $c = 0$ and $y^* > 0$, the above result yields that $y(t) > 0$ for all $t \geq T$.

Now let us prove part (i). Since $0 < y(T) < h^{-1}(c)$, then either $0 < y(t) < h^{-1}(c)$ for any $t \geq T$ and we are done, or there exists a $t_2 > T$ such that $0 < y(t) < h^{-1}(c)$ for $0 < t < t_2$ and either $y(t_2) = 0$ or $y(t_2) = h^{-1}(c)$. But this later case is not possible, since $y(t) \neq h^{-1}(c)$ for all $t \geq T$. If $y(t_2) = 0$, then one can easily see that

$\dot{y}(t_2) \leq 0$. On the other hand, we get by $(\mathbf{H_1})$, $(\mathbf{H_2})$ and from (3.2.3) that

$$\dot{y}(t_2) = r(t_2)[c - h(y(t_2))] = r(t_2)[c - h(0)] = cr(t_2) > 0,$$

which is a contradiction. Hence $0 < y(t) < h^{-1}(c)$ for any $t \geq T$, and therefore $\dot{y}(t) > 0$. Since $y(t)$ is bounded, the solution $y(t)$ exists for all $t \geq T$, and since it is monotone increasing, $y(t)$ has a finite limit at ∞, and

$$N := \lim_{t\to\infty} y(t) \leq h^{-1}(c).$$

We show that $N = h^{-1}(c)$. Otherwise $N < h^{-1}(c)$, in this case since $\dot{y}(t) > 0$, by integrating (3.2.3) from T to t we get

$$y(t) = y(T) + \int_T^t r(s)[c - h(y(s))]ds \geq y(T) + \int_T^t r(s)[c - h(N)]ds,$$

and as $t \to \infty$ we have by $(\mathbf{H_1})$ that

$$N \geq y(T) + [c - h(N)] \int_T^\infty r(s)ds = \infty.$$

This contradicts with the boundedness of $y(t)$, and hence

$$N = h^{-1}(c).$$

Now we prove part (ii). If $y(T) = h^{-1}(c)$, then it is clear that $y(t) = h^{-1}(c)$ is an equilibrium solution of (3.2.3) and (3.2.4), and it is easy to argue that $y(t) = h^{-1}(c)$ is the unique solution in this case.

The proof of part (iii) is similar to the proof of part (i), so it is omitted here.

Finally, we show the uniqueness of the solution. Let $T \geq 0$, $y^* > 0$ and $c \geq 0$ be fixed. Suppose both y_1 and y_2 satisfy the corresponding IVP (3.2.3) with (3.2.4). It follows from properties (i)–(iii) that both solutions exist on $[T, \infty)$, and $y_1(t) > 0$ and $y_2(t) > 0$ for all $t \geq T$. Suppose there exist $t_2 > T$ such that $y_1(t_2) > y_2(t_2)$ (the opposite case can be treated similarly). Then there exists $t_1 \in [T, t_2)$ such that $y_1(t_1) = y_2(t_1)$ and $y_1(t) > y_2(t)$ for $t \in (t_1, t_2)$. Define $z(t) := y_1(t) - y_2(t)$. Then z is continuously differentiable, $z(t_1) = 0$, $z(t) > 0$ for $t \in (t_1, t_2)$. On the other hand, Eq. (3.2.3) and the strict monotonicity of h imply

$$\dot{z}(t) = \dot{y}_1(t) - \dot{y}_2(t) = r(t)\Big(h(y_2(t)) - h(y_1(t))\Big) < 0, \quad t \in (t_1, t_2),$$

which is a contradiction. This yields that $y_1(t) = y_2(t)$ must hold for $t > T$. $\square$

A.2 Proofs of some results in Chapter 5

Proof of Lemma 5.2.3. The proof of part (i) is obtained directly from Theorem 4.2.1, where we can rewrite (5.2.16) in the form (4.2.1) with $\gamma_i(u) := f_i(u) - m_{ii}h_{ii}(u) - l_i$ and $g_{ij}(u) := m_{ij}h_{ij}(u)$ for each $1 \le i \neq j \le n$ and $g_{ii}(u) = 0$. Now, to prove the existence of a positive solution for System (5.2.16), we check that conditions **(A)** and **(B)** of Theorem 4.2.1 are satisfied. For condition **(A)**, we have that $\gamma_i(u) = 0$ if and only if

$$\frac{f_i(u)}{h_{ii}(u)} = \frac{l_i}{h_{ii}(u)} + m_{ii}, \qquad 1 \le i \le n. \tag{A.2.1}$$

The left hand side of (A.2.1) is increasing and the right hand side of (A.2.1) is decreasing, moreover, either the left hand side or the right hand side is a strictly monotone function. Therefore, condition **(A)** of Theorem 4.2.1 holds, if we show

$$\lim_{u\to 0^+} \frac{f_i(u)}{h_{ii}(u)} < \lim_{u\to 0^+} \frac{l_i}{h_{ii}(u)} + m_{ii}, \qquad 1 \le i \le n, \tag{A.2.2}$$

and

$$\lim_{u\to \infty} \frac{f_i(u)}{h_{ii}(u)} > \lim_{u\to \infty} \frac{l_i}{h_{ii}(u)} + m_{ii}, \qquad 1 \le i \le n. \tag{A.2.3}$$

If $l_i > 0$ and $h_{ii}(0) = 0$, then (A.2.2) follows, since the left hand side of (A.2.2) is always finite, since $\frac{f_i(u)}{h_{ii}(u)}$ is monotone increasing. If $l_i > 0$ and $h_{ii}(0) > 0$, then the right hand side of (A.2.2) is finite and positive, but $\lim\limits_{u\to 0^+} \frac{f_i(u)}{h_{ii}(u)} = 0$ using $(\mathbf{A_3})$. If $l_i = 0$, then assumption (5.2.17) yields (A.2.2). Relation (A.2.3) follows immediately from (5.2.18). Hence condition **(A)** is satisfied.

To check condition **(B)**, we see that $g_{ij}(u) := m_{ij}h_{ij}(u)$, $1 \le i \neq j \le n$, and $g_{ii}(u) = 0$ are increasing on $\mathbb{R}_+$, and relation (4.2.3) is equivalent to

$$\sum_{\substack{j=1 \\ j\neq i}}^{n} m_{ij}h_{ij}(u) < f_i(u) - m_{ii}h_{ii}(u) - l_i,$$

which is satisfied if and only if

$$\sum_{j=1}^{n} m_{ij}\frac{h_{ij}(u)}{f_i(u)} + \frac{l_i}{f_i(u)} < 1.$$

Therefore, using (5.2.18), (4.2.3) is satisfied when u is large enough and hence condition **(B)** is satisfied. Therefore (5.2.16) has a positive solution. For the proof of the uniqueness of the positive solution of the System (5.2.16), we check that conditions **(C)** and **(D)** of Theorem 4.2.1 are satisfied. Since $m_{ij} \geq 0$ and $h_{ij}(u) > 0$ for $u > 0$, for each $1 \leq i, j \leq n$, then condition **(C)** is satisfied. To check condition **(D)**, suppose $m_{ij} > 0$. Then the function

$$\begin{aligned}\frac{\gamma_j(u)}{g_{ij}(u)} &= \frac{f_j(u) - m_{jj}h_{jj}(u) - l_j}{m_{ij}h_{ij}(u)} \\ &= \frac{f_j(u)}{m_{ij}h_{ij}(u)} - \frac{m_{jj}h_{jj}(u)}{m_{ij}h_{ij}(u)} - \frac{l_j}{m_{ij}h_{ij}(u)}\end{aligned}$$

is monotone increasing on $(0, \infty)$, by $(\mathbf{A_4})$ and $(\mathbf{C_1})$. By assumption $(\mathbf{C_3})$, there exists $i \neq j$ such that $\frac{\gamma_j(u)}{g_{ij}(u)}$ is strictly monotone increasing on $(0, \infty)$, and so condition **(D)** is satisfied. Hence the System (5.2.16) has a unique positive solution.

Now we prove (ii). From (5.2.19) we have

$$x_i \geq f_i^{-1}\left(\sum_{j=1}^{n} m_{ij}h_{ij}(x_j) + l_i\right), \qquad 1 \leq i \leq n. \tag{A.2.4}$$

Assumption $(\mathbf{A_3})$ and (5.2.17) yield that there exists a small u^* such that

$$0 < u^* < x_i, \qquad 1 \leq i \leq n. \tag{A.2.5}$$

and

$$1 \leq \sum_{j=1}^{n} m_{ij}\frac{h_{ij}(u^*)}{f_i(u^*)} + \frac{l_i}{f_i(u^*)},$$

or equivalently,

$$0 < u^* \leq f_i^{-1}\left(\sum_{j=1}^{n} m_{ij}h_{ij}(u^*) + l_i\right), \qquad 1 \leq i \leq n. \tag{A.2.6}$$

Now we construct a sequence $(x_i^{(0)}, ..., x_i^{(k)}, ...)$ such that

$$x_i^{(0)} = u^* \text{ and } x_i^{(k+1)} = f_i^{-1}\left(\sum_{j=1}^{n} m_{ij}h_{ij}(x_j^{(k)}) + l_i\right), \quad k \geq 0, \qquad 1 \leq i \leq n, \tag{A.2.7}$$

and we prove that the sequence $(x_i^{(0)}, ..., x_i^{(k)}, ...)$ converges. For this, we prove that the sequence $(x_i^{(0)}, ..., x_i^{(k)}, ...)$ is monotone increasing and bounded from above. First we show

$$x_i^{(k+1)} \geq x_i^{(k)}, \qquad \text{for all} \qquad k \geq 0, \qquad 1 \leq i \leq n. \tag{A.2.8}$$

For this aim, we use the mathematical induction, so at $k = 0$ we have, by (A.2.6) and (A.2.7),

$$x_i^{(1)} = f_i^{-1}\left(\sum_{j=1}^{n} m_{ij}h_{ij}(x_j^{(0)}) + l_i\right) = f_i^{-1}\left(\sum_{j=1}^{n} m_{ij}h_{ij}(u^*) + l_i\right) \geq u^* = x_i^{(0)}, \quad 1 \leq i \leq n.$$

Next, we assume, for some $k \geq 0$, that

$$x_i^{(k)} \geq x_i^{(k-1)}, \qquad 1 \leq i \leq n. \tag{A.2.9}$$

Then, by (A.2.7) and (A.2.9),

$$x_i^{(k+1)} = f_i^{-1}\left(\sum_{j=1}^{n} m_{ij}h_{ij}(x_j^{(k)}) + l_i\right) \geq f_i^{-1}\left(\sum_{j=1}^{n} m_{ij}h_{ij}(x_j^{(k-1)}) + l_i\right) = x_i^{(k)}, \quad 1 \leq i \leq n.$$

Hence the sequence $(x_i^{(0)}, ..., x_i^{(k)}, ...)$ is monotone increasing for all $k \geq 0, \ \ 1 \leq i \leq n$. Now to prove that the sequence $(x_i^{(0)}, ..., x_i^{(k)}, ...)$ is bounded from above for all $k \geq 0, \ \ 1 \leq i \leq n$, we show that

$$x_i^{(k+1)} \leq x_i, \qquad \text{for all} \qquad k \geq 0, \qquad 1 \leq i \leq n. \tag{A.2.10}$$

Again we use the mathematical induction, so at $k = 0$ we have, by (A.2.4), (A.2.5) and (A.2.7),

$$\begin{aligned} x_i^{(1)} &= f_i^{-1}\left(\sum_{j=1}^{n} m_{ij}h_{ij}(x_j^{(0)}) + l_i\right) \\ &= f_i^{-1}\left(\sum_{j=1}^{n} m_{ij}h_{ij}(u^*) + l_i\right) \\ &\leq f_i^{-1}\left(\sum_{j=1}^{n} m_{ij}h_{ij}(x_j) + l_i\right) \\ &\leq x_i, \qquad\qquad 1 \leq i \leq n. \end{aligned}$$

Next, we assume, for some $k \geq 0$, that

$$x_i^{(k)} \leq x_i, \qquad 1 \leq i \leq n. \tag{A.2.11}$$

Then, by (A.2.4), (A.2.7) and (A.2.11), we have

$$\begin{aligned} x_i^{(k+1)} &= f_i^{-1}\left(\sum_{j=1}^{n} m_{ij}h_{ij}(x_j^{(k)}) + l_i\right) \\ &\le f_i^{-1}\left(\sum_{j=1}^{n} m_{ij}h_{ij}(x_j) + l_i\right) \\ &\le x_i, \qquad 1 \le i \le n, \end{aligned}$$

and hence the sequence $(x_i^{(0)}, ..., x_i^{(k)}, ...)$ is bounded from above for all $k \ge 0,\ 1 \le i \le n$. Now since the sequence is monotone increasing and bounded from above, then it converges and has a finite limit, i.e.,

$$\lim_{k\to\infty} x_i^{(k)} = x_i^*, \qquad 1 \le i \le n,$$

and clearly, $x^* = (x_1^*, ..., x_n^*)$ is the unique positive solution of (5.2.16). On the other hand, we know that

$$x_i^{(k)} \le x_i, \qquad k \ge 0, \qquad 1 \le i \le n,$$

which implies

$$x_i^* \le x_i, \qquad 1 \le i \le n,$$

and hence the proof of (ii) is completed.

The proof of part (iii) is similar to that of part (ii), so it is omitted here. □

Proof of Theorem 5.2.4. In the proof we will use the notations

$$\underline{x}_i^{\varphi}(\infty) := \liminf_{t\to\infty} x_i(\varphi)(t) \quad \text{and} \quad \overline{x}_i^{\varphi}(\infty) := \limsup_{t\to\infty} x_i(\varphi)(t).$$

By conditions (5.2.4), (5.2.5), (5.2.7) and relation (5.2.10), we have for any $T \ge \tau$ that

$$0 \le m_{ij}(T) := \inf_{t\ge T} \frac{\sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t)}{r_i(t)} \le \sup_{t\ge T} \frac{\sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t)}{r_i(t)} =: M_{ij}(T) < \infty, \quad 1 \le i,j \le n; \tag{A.2.12}$$

$$0 \le l_i(T) := \inf_{t\ge T} \frac{\rho_i(t)}{r_i(t)} \le \sup_{t\ge T} \frac{\rho_i(t)}{r_i(t)} =: L_i(T) < \infty, \qquad 1 \le i \le n; \tag{A.2.13}$$

and

$$0 < \underline{x}_i(T) := \inf_{t \geq T-\tau} x_i(t) \leq \sup_{t \geq T-\tau} x_i(t) =: \overline{x}_i(T) < \infty, \qquad 1 \leq i \leq n. \qquad \text{(A.2.14)}$$

Thus from (A.2.12), (A.2.13), (A.2.14) in (5.2.1) we get

$$\begin{aligned}
\dot{x}_i(t) &\geq r_i(t) \left[\sum_{j=1}^{n} \frac{\sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t)}{r_i(t)} h_{ij}(\underline{x}_j(T)) + l_i(T) - f_i(x_i(t)) \right] \\
&\geq r_i(t) \left[\sum_{j=1}^{n} \inf_{t \geq T} \frac{\sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t)}{r_i(t)} h_{ij}(\underline{x}_j(T)) + l_i(T) - f_i(x_i(t)) \right] \\
&\geq r_i(t) \left[\sum_{j=1}^{n} m_{ij}(T) h_{ij}(\underline{x}_j(T)) + l_i(T) - f_i(x_i(t)) \right], \qquad t \geq T,\ 1 \leq i \leq n,
\end{aligned}$$

or equivalently

$$\dot{x}_i(t) \geq r_i(t) \left[C_i(T) - f_i(x_i(t)) \right], \qquad t \geq T, \qquad 1 \leq i \leq n, \qquad \text{(A.2.15)}$$

where $C_i(T) := \sum_{j=1}^{n} m_{ij}(T) h_{ij}(\underline{x}_j(T)) + l_i(T)$. From (A.2.15) and the comparison theorem of differential inequalities we get

$$x_i(t) \geq y_i(t), \qquad t \geq T, \quad 1 \leq i \leq n,$$

where $y_i(t) = y(T, \varphi_i(T), C_i(T), r_i, f_i)(t)$, $1 \leq i \leq n$ are the solutions of the differential equations

$$\dot{y}(t) = r_i(t) \Big(c - f_i(y(t)) \Big), \qquad t \geq T \geq 0, \qquad \text{(A.2.16)}$$

with $c = C_i(T)$ and with the initial condition

$$y_i(T) = x_i(T), \qquad 1 \leq i \leq n. \qquad \text{(A.2.17)}$$

So, from Lemma 3.2.1, we see that

$$\lim_{t \to \infty} y_i(t) = f_i^{-1} \left(C_i(T) \right), \qquad 1 \leq i \leq n.$$

Thus, for any $T \geq \tau$,

$$\underline{x}_i^{\varphi}(\infty) := \liminf_{t \to \infty} x_i(\varphi)(t) \geq \lim_{t \to \infty} y_i(t) = f_i^{-1} \left(C_i(T) \right), \qquad 1 \leq i \leq n.$$

But

$$\begin{aligned}\lim_{T\to\infty} f_i^{-1}\left(C_i(T)\right) &= \lim_{T\to\infty} f_i^{-1}\left(\sum_{j=1}^{n} m_{ij}(T)h_{ij}(\underline{x}_j(T)) + l_i(T)\right)\\ &= f_i^{-1}\left(\sum_{j=1}^{n} \lim_{T\to\infty} m_{ij}(T)h_{ij}(\underline{x}_j(T)) + \lim_{T\to\infty} l_i(T)\right)\\ &= f_i^{-1}\left(\sum_{j=1}^{n} \underline{m}_{ij}h_{ij}(\underline{x}_j^{\varphi}(\infty)) + \underline{l}_i\right), \qquad 1\le i\le n.\end{aligned}$$

Therefore

$$\underline{x}_i^{\varphi}(\infty) \ge f_i^{-1}\left(\sum_{j=1}^{n} \underline{m}_{ij}h_{ij}(\underline{x}_j^{\varphi}(\infty)) + \underline{l}_i\right), \qquad 1\le i\le n,$$

or equivalently

$$f_i(\underline{x}_i^{\varphi}(\infty)) \ge \sum_{j=1}^{n} \underline{m}_{ij}h_{ij}(\underline{x}_j^{\varphi}(\infty)) + \underline{l}_i, \qquad 1\le i\le n.$$

Since all the conditions of Lemma 5.2.3 are satisfied with $m_{ij} = \underline{m}_{ij}$ and $l_i = \underline{l}_i$, it can be applied, and we obtain

$$\underline{x}_i^{\varphi}(\infty) \ge \underline{x}_i^*, \qquad 1\le i\le n,$$

where $\underline{x}^* = (\underline{x}_1^*, \ldots, \underline{x}_n^*)$ is the unique positive solution of the System (5.2.25). In a similar way we can get

$$\overline{x}_i^{\varphi}(\infty) \le \overline{x}_i^*, \qquad 1\le i\le n,$$

where $\overline{x}^* = (\overline{x}_1^*, \ldots, \overline{x}_n^*)$ is the unique positive solution of the System (5.2.26). Hence the proof is completed. □

Proof of Theorem 5.3.3. Let $\varphi, \psi \in C_+^n$ be fixed and define $\nu_i(t) := x_i(\varphi)(t)$ and $\omega_i(t) := x_i(\psi)(t)$. Then

$$\dot{\nu}_i(t) = \sum_{j=1}^{n}\sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t)\nu_j(t-\tau_{ij\ell}(t)) - r_i(t)\nu_i^{q_i}(t) + \rho_i(t), \qquad t\ge 0,\ 1\le i\le n,$$

and

$$\dot{\omega}_i(t) = \sum_{j=1}^{n}\sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t)\omega_j(t-\tau_{ij\ell}(t)) - r_i(t)\omega_i^{q_i}(t) + \rho_i(t), \qquad t\ge 0,\ 1\le i\le n.$$

Now, introduce $z_i(t) := \nu_i(t) - \omega_i(t)$, then

$$\dot{z}_i(t) = \sum_{j=1}^{n}\sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t) z_j(t-\tau_{ij\ell}(t)) - r_i(t) z_i(t) \sum_{r=0}^{q_i-1} \nu_i^r(t)\omega_i^{q_i-1-r}(t), \quad t \geq 0,\ 1 \leq i \leq n,$$

or equivalently

$$\dot{z}_i(t) = -a_i(t) z_i(t) + \sum_{j=1}^{n}\sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t) z_j(t-\tau_{ij\ell}(t)), \qquad t \geq 0,\ \ 1 \leq i \leq n, \quad \text{(A.2.18)}$$

where $a_i(t) := r_i(t) \sum\limits_{r=0}^{q_i-1} \nu_i^r(t)\omega_i^{q_i-1-r}(t)$. We can consider (A.2.18) as the perturbation of the scalar ordinary differential equation

$$\dot{y}_i(t) = -a_i(t) y_i(t), \qquad t \geq 0,\ \ 1 \leq i \leq n.$$

Thus, for any $T \geq 0$ and $1 \leq i \leq n$, the solution of (A.2.18) satisfies

$$z_i(t) = z_i(T) e^{-\int_T^t a_i(u)du} + \int_T^t e^{-\int_s^t a_i(u)du} \sum_{j=1}^{n}\sum_{\ell=1}^{n_0} \alpha_{ij\ell}(s) z_j(s-\tau_{ij\ell}(t))\, ds, \quad t \geq T. \tag{A.2.19}$$

The definition of $a_i(t)$, (5.3.13) and assumption (5.3.14) yield, for each $i = 1, \ldots, n$,

$$\begin{aligned}
\limsup_{t\to\infty} \frac{\sum\limits_{j=1}^{n}\sum\limits_{\ell=1}^{n_0} \alpha_{ij\ell}(t)}{a_i(t)} &\leq \limsup_{t\to\infty} \frac{1}{\sum\limits_{r=0}^{q_i-1} \nu_i^r(t)\omega_i^{q_i-1-r}(t)} \limsup_{t\to\infty} \frac{\sum\limits_{j=1}^{n}\sum\limits_{\ell=1}^{n_0} \alpha_{ij\ell}(t)}{r_i(t)} \\
&\leq \frac{1}{q_i \underline{m}_{ii}} \sum_{j=1}^{n} \limsup_{t\to\infty} \frac{\sum\limits_{\ell=1}^{n_0} \alpha_{ij\ell}(t)}{r_i(t)} \\
&\leq \frac{\sum\limits_{j=1}^{n} \overline{m}_{ij}}{q_i \underline{m}_{ii}} \\
&< 1.
\end{aligned}$$

Thus, there exist $0 < \eta < 1$ and $T_1 \geq 0$ such that

$$\frac{\sum\limits_{j=1}^{n}\sum\limits_{\ell=1}^{n_0} \alpha_{ij\ell}(t)}{a_i(t)} < \eta < 1, \qquad t \geq T_1,$$

or equivalently

$$\sum_{j=1}^{n}\sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t) \leq \eta a_i(t), \qquad t \geq T_1, \quad 1 \leq i \leq n. \tag{A.2.20}$$

We introduce $\overline{z}_j(\infty) := \limsup\limits_{t\to\infty} z_j(t)$, $1 \leq j \leq n$. For every $\varepsilon > 0$, there exists a

$T \geq T_1$ such that

$$|z_j(s-\tau_{ij\ell}(t))| \leq \overline{z}_j(\infty)+\varepsilon \leq \max_{1\leq l\leq n} \overline{z}_l(\infty)+\varepsilon, \qquad s \geq T, \quad 1 \leq i,j \leq n, 1 \leq \ell \leq n_0. \tag{A.2.21}$$

Using (A.2.19), (A.2.20) and (A.2.21), we get

$$\begin{aligned}
|z_i(t)| &\leq |z_i(T)|e^{-\int_T^t a_i(u)\,du} + \int_T^t e^{-\int_s^t a_i(u)\,du} \sum_{j=1}^{n} \sum_{\ell=1}^{n_0} \alpha_{ij\ell}(s)|z_j(s-\tau_{ij\ell}(t))|\,ds \\
&\leq |z_i(T)|e^{-\int_T^t a_i(u)\,du} + \Big(\max_{1\leq j\leq n} \overline{z}_j(\infty)+\varepsilon\Big)\eta \int_T^t e^{-\int_s^t a_i(u)\,du} a_i(s)\,ds \\
&= |z_i(T)|e^{-\int_T^t a_i(u)\,du} + \Big(\max_{1\leq j\leq n} \overline{z}_j(\infty)+\varepsilon\Big)\eta(1-e^{-\int_T^t a_i(u)\,du})
\end{aligned}$$

for $t \geq T$ and $1 \leq i \leq n$. Taking the limit supermum for both sides as $t \to \infty$, and using $(\mathbf{A_1})$ and Lemma 5.2.2, and that

$$\begin{aligned}
\int_T^\infty a_i(u)du &= \int_T^\infty r_i(u) \sum_{r=0}^{q_i-1} \nu_i^r(u)\omega_i^{q_i-1-r}(u)du \\
&\geq \left(\inf_{t\geq T} \sum_{r=0}^{q_i-1} \nu_i^r(t)\omega_i^{q_i-1-r}(t)\right) \int_T^\infty r_i(u)du \\
&= \infty,
\end{aligned}$$

we obtain

$$\overline{z}_i(\infty) \leq \eta(\max_{1\leq l\leq n} \overline{z}_l(\infty)+\varepsilon), \qquad 1 \leq i \leq n.$$

Thus

$$\max_{1\leq i\leq n} \overline{z}_i(\infty) \leq \eta \max_{1\leq i\leq n} \overline{z}_i(\infty) + \eta\varepsilon,$$

which implies

$$\max_{1\leq i\leq n} \overline{z}_i(\infty) \leq \frac{\eta\varepsilon}{1-\eta}.$$

Since $\varepsilon > 0$ can be arbitrary small, we get $\max_{1\leq i\leq n} \overline{z}_i(\infty) = 0$ and consequently $\lim_{t\to\infty} z_i(t) = 0$, $1 \leq i \leq n$. Hence the proof is completed. $\square$