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Vibrations and Stability of Heterogeneous Curved Beams

Synopsis of PhD theses by
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1 Preliminaries

In recent decades, curved beams have been widely used in numerous engineering applications as load carrying members. Let us just think about arch bridges, roof structures or stiffeners in the aerospace or marine industry. Scientists and designers are always being interested in the mechanical behaviour (stresses, displacements, load carrying capabilities, etc.) of such structural elements to prevent failure (e.g.: yielding, buckling, self-excited vibrations) under given loads and circumstances. Therefore, there are a number of books, articles and other scientific works delivering relevant results, see, e.g., [1,2,3] for calculating the stresses, [4,5,6] for stability problems and [6,7,8] as regards the issue of vibrations.

Nowadays not only homogeneous members but inhomogeneous or heterogeneous ones are also getting more and more widespread. These beams can have more advantageous properties compared to the homogeneous ones, such as reduced weight; improved corrosion, fatigue and chemical resistance and higher strength. A class of nonhomogeneous material composition is the so-called cross-sectional inhomogeneity. It means that the material parameters – say, Young's modulus E or the Poisson ratio ν – have symmetric distribution with respect to the cross-sectional axis ζ. This distribution is either continuous or constant over each segment (layer). Some illustrative examples are shown in Figure 1. In this way it is possible to simply model multilayered or functionally graded materials. For planar, elastic, isotropic circular beams of this kind, I intend to focus on three mechanical issues: stresses, stability and vibrations.
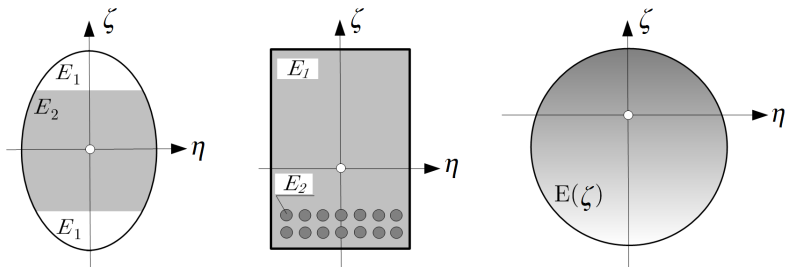

Figure 1: The concept of cross-sectional inhomogeneity.

As regards the mechanical behaviour of curved beams, investigations began in the 19th century. The foremost load-displacement relationship was established by Bresse (1854). Winkler was the first to derive a formula for the normal stress distribution (1858) and Grashof is known for developing an equilibrium method (1878) for the calculation of the shear stresses. These results are well collected in the works [1,9].

The interest is still live, as new models for different loading cases, geometries, and even for nonhomogeneous materials are continuously being published. For instance, Ascione and Fraternali [10] use the finite element method to obtain solutions for the stresses in perfectly bonded layered curved beams. They assume that each layer is a Timoshenko beam. They compute interlaminar, normal and shear stresses as well. Segura and Armengaud [11] propose simple analytical solutions for the normal and shearing stress distributions in composites under bending loads. The normal stress distribution due to the bending moment and the axial force is hyperbolic over the cross-section. The authors have also managed to extend Bredt's formula for composite curved beams to get the shear stresses. Article [12] by Baksa and Ecsedi provides formulae for the stress distributions in straight beams with cross-sectional inhomogeneity under pure bending. Book [1] by Kozák and Szeidl also deserves mentioning as it presents how to derive the stresses in straight beams with cross-sectional inhomogeneity and also gathers formulae for the stress distributions in homogeneous curved beams. According to the literature review, it seems that curved beams with cross-sectional inhomogeneity have not yet been investigated.

Another popular topic is the buckling behaviour of beams. In 1757, Euler derived his well-known formula for the critical (buckling) load of straight bars under compression. Considering the behaviour of curved members, stability investigations began much later: around the beginning of the 19th century. The early literature ignored the extensibility of the centerline – see, e.g., [13] by Hurlbrink. Then Chwalla and Kollbrunner [14] showed that account for the axial strain can notably affect the critical load. After the 1950s, work became more intensive. Szeidl in his PhD thesis [6] determines the critical load of circular beams under radial dead load given that the Fourier series of the load is known. Paper [15] by DaDeppo and Schmidt provides solution to the buckling load of deep circular beams whose loading is a vertical force. The authors have shown that quadratic terms should be accounted in the analysis.

When dealing with shallow circular beams Pi, Bradford et al. have pointed out [4,16] that account even for the pre-buckling deformations is likewise essential not to overestimate the permissible load. The authors have been intensively investigating the stability of homogeneous (shallow and deep) arches using their analytical model, which accounts for all the above mentioned properties. Nonlinearities are considered through the square of the infinitesimal rotations. The authors have evaluated their model for various loads (distributed, concentrated) and boundary conditions (pinned, fixed, elastic supports, mixed supports, etc.). Bateni et al. [5] use the same kinematical hypotheses as presented in [4] to analyse shallow arches under a concentrated load. However, their model is valid for functionally graded materials.

The vibrations of curved beams has been a field of interest as of the 1920s. Den Hartog was the first to investigate the free vibrations of such structural elements (1928). Early but still notable contributions − assuming the inextensibility of the centerline − were provided in [17,18].

Szeidl in his PhD thesis [6] investigates how the extensibility of the centerline can change the eigenfrequencies of the free vibrations of planar circular beams under a constant radial load. The author achieves results using the Green function matrix, with what, the related boundary value problem is transformed to a problem governed by Fredholm integral equations. Kang et al. [19] obtain the frequencies (eigenvalues) for the in- and out-of-plane vibrations of circular Timoshenko arches given that rotatory inertia and shear deformations are accounted. Tüfekçi and Arpaci [7] managed to gain exact analytical solutions for the in-plane free harmonic vibrations of circular arches. The authors account for the extensibility of the centerline and also for the transverse shear and rotatory inertia effects. Kovács [8] deals with layered arches assuming the possibility of both perfect and even imperfect bonding between any two nearby layers.

In the reviewed literature there are some sources, which use the Green function to tackle some dynamic issues. Szeidl et al. [20] determine the natural frequencies of pinned and fixed circular arches using this technique. Kelemen [21] extends the former model. She computes the natural frequencies as functions of a constant distributed radial load. Li et al. [22] consider the forced vibrations of straight Timoshenko beams when these are under a time harmonic concentrated load. Damping effects at the ends are accounted.

2 Objectives

Based on the reviewed open literature, the main objectives of the candidate are related to cross-sectional inhomogeneity and are detailed in the forthcoming.

Objective 1: Generalization of some classical results valid for homogeneous materials. These investigations are aimed to lead to the following results:

- Generalization of two elementary relationships (valid for homogeneous curved beams) − that provide the normal stress caused by an axial force and a bending moment − for curved beams with cross-sectional inhomogeneity.
- Setting up a further formula for computing the shear stress.
- In addition, a formula for the shear correction factor is also to be derived.

– The results obtained for the stresses should be compared with those obtained by finite element computations.

Objective 2: On the basis of the literature overview, no investigations have been carried out concerning the stability problem of (shallow) circular beams under the assumption of cross-sectional inhomogeneity. Within the frames of what has been mentioned above, Objective 2 is summarized in the following items.

– I intend to develop a new nonlinear model for non-strictly shallow curved beams from the principle of virtual work. It is aimed to be more accurate than those presented in [4,23] and should be applicable to cross-sectional inhomogeneity as well.

– I would like to evaluate the new model for pinned-pinned, fixed-fixed and rotationally restrained supports provided that the beam is subjected to a central load at the crown point. This would involve the determination of the critical load both for symmetric snap-through and antisymmetric bifurcation buckling.

– At the same time, the typical buckling ranges and its endpoints are also of interest.

– Comparison of the results with those available in the literature and with the Abaqus commercial finite element software should also be performed.

Objective 3 is related to the in-plane vibrations of loaded circular beams with cross-sectional inhomogeneity. I intend

– to derive those boundary value problems, which can make it clear how a radial load affects the natural frequencies of pinned-pinned and fixed-fixed beams,

– to construct the corresponding Green function matrices by taking into account that the central load at the crown point can either be compressive or tensile (four Green function matrices are to be determined),

– to reduce the eigenvalue problems set up for the natural frequencies (which depend on the load) to eigenvalue problems governed by homogeneous Fredholm integral equation systems (four systems should be established),

– to replace these eigenvalue problems with algebraic ones and to solve them numerically,

– to clarify how the vertical force at the crown point affects the frequencies of the vibrations (when this load is removed, I have to get back the results valid for free vibrations),

– to verify some results by Abaqus and/or experiments.

3 Investigations performed

While establishing the mechanical models, the validity of the following common hypotheses were considered:

- there is cross-sectional inhomogeneity,
- the displacements and deformations are sufficiently small,
- the beam models are one-dimensional,
- the (E-weighted) centerline remains in its own plane,
- the curved beam has uniform cross-section and constant initial radius,
- the cross-section is symmetric,
- the classical single-layer theory applies,
- the magnitude of the normal stress σ_ξ is much greater than that of the stress components σ_η and σ_ζ.

When deriving simple closed-form solutions for the normal stress distribution, the validity of the Euler-Bernoulli theory is assumed. Such loads that cause bending action and axial strain can be applied (with shearing effects neglected). First, an exact formula is derived. Then further transformations and simplifications lead to the generalized form of the Grashof (Winkler) formula. Accordingly, the bending moment has a constant and hyperbolic effect on the normal stress distribution while the axial force causes constant stress. A further achievement is another formula for the normal stress and for the location of the neutral axis in the case of pure bending – both are dependent on the material composition.

The shear stresses are obtained by using equilibrium equations for a portion of the beam (i.e., the kinematical relations are not completely satisfied). The result is the extension of Grashof's equilibrium method for cross-sectional inhomogeneity. The advantage of this procedure is the relatively simple outcome. Moreover, a formula is proposed for the shear correction factor.

The static stability model is based on the Euler-Bernoulli hypothesis. The kinematical assumption contains a quadratic term, that is, the square of the infinitesimal rotations. Given that the investigated structural element is primarily a shallow arch, the effect of the tangential displacements on the former quantity is neglected. As the pre-buckling deformations are substantial, the change in the equilibrium state due to the deformations is accounted. The governing equilibrium equations under concentrated and distributed loads for non-uniform rotational end restraints are established using the principle of virtual work. However, solution is calculated only when there is a concentrated dead load exerted at the crown point. Due to the symmetry properties, a half-beam model is examined. The nonlinear axial strain on the centerline is constant under these conditions. In this way, a fourth-order ordinary differential equation governs the problem mathematically, which can be solved

in closed-form. The former statements are valid even for the incremental quantities, which are measured after the loss of stability.

Semi-analytical evaluations are carried out for symmetrically supported pinned, fixed and rotationally restrained circular beams. These evaluations, on the one hand, include the determination of the pre-buckling equilibrium in terms of the material, geometry and loading. On the other hand, it is also pointed out that there are two possible buckling modes: antisymmetric bifurcation buckling with no strain increment and symmetric snap-through mode with a changing centerline length. The related critical strains and therefore the critical loads are determined in terms of the geometry. It is found that there are beams for which there is no buckling. As for the others it is also sought which of the two buckling modes dominates in terms of the geometry.

For pinned beams, mostly antisymmetric buckling can be expected. However, for fixed ones the symmetric type governs. When the spring stiffness of the supports is (equal to zero) [tends to infinity] we get back the solutions valid for (pinned) [fixed] beams. To better understand the behaviour of the members, the primary equilibrium paths are also plotted for each typical buckling range. Commercial finite element computations and comparison with the literature indicate that the results can be considered as valid for all checked supports and even for not strictly shallow arches. Simple numerical examples show that material heterogeneity can have a significant impact on the permissible loads, therefore account for this property seems inevitable.

The vibration analysis is based on linearized strains and the Euler-Bernoulli hypothesis. At the same time, the effect of the tangential displacements on the rigid body rotations are kept so that the results are applicable for deep arches as well. The natural frequencies are sought and that how a central concentrated load changes these frequencies. The equilibrium equations are derived from the principle of virtual work for a beam under concentrated and distributed loads. The strain the concentrated load causes is constant on the centerline. The pre-buckling (initial) equilibrium is governed by ordinary differential equations.

As for the dynamic part of the issue, the forces of inertia are accounted and undamped time harmonic vibrations are considered. The derivations lead to an eigenvalue problem where the square of the eigenfrequencies are proportional to the eigenvalues. Solutions are sought for those cases when the central vertical concentrated force causes compression and tension.

The Green function matrix is constructed in closed-form for both loading cases of pinned and fixed beams. The application of this technique requires linear ordinary differential equations with closed-form general solutions and self-adjoint eigenvalue problems. With the corresponding Green functions in hand, each eigenvalue problem governed by ordinary differential equations and the corresponding boundary conditions can be replaced by homogeneous

Fredholm integral equations and following the procedure presented in [6], they can numerically be reduced to algebraic equation systems (eigenvalue problems).

When dealing with the vibrations, we must also be aware of the critical loads because if this limit is reached, buckling occurs. So these loads are also determined. Since in practise, the load is the known quantity and the model has the strain as parameter, a unique relationship between these quantities is provided.

Results are evaluated both for the free and loaded vibrations and are compared with the literature and commercial finite element software computations. Moreover, colleagues from Romania contributed with some measurements regarding the free vibrations of fixed beams. Thanks to their efforts it became possible to compare some numerical results also with experiments to confirm the validity of the model.

Regarding the outcomes, the quotients of the even unloaded frequencies of curved and straight beams with the same length and material only depend on the central angle and the supports, while the odd ones are also functions of the cross-sectional geometry and material distribution. It turns out that for pinned beams the quotient of the square of the second loaded and unloaded frequencies (increase) [decrease] almost linearly under (tension) [compression] in terms of the strain-critical strain ratio, meanwhile the central angle, geometry and material do not affect these relations. The experiences are more likely quadratic and more dependent on the geometry for fixed members. The effect of the material composition on the frequencies is illustrated through simple numerical examples.

4   Novel results

The first objective was to provide simple formulae for calculating the stress state of heterogeneous curved beams by generalizing the formulae valid for homogeneous curved beams. These involved the expressions of the normal stress and shear stress. The shear correction factor was also determined. The most important results are gathered in

STATEMENT 1.

1.a.   I have derived an exact and two approximative relationships that provide the normal stress caused by an axial force and a bending moment in curved beams with cross-sectional inhomogeneity. The latter two are generalizations of well-known relationships valid for homogeneous curved beams. A further formula has been established for computing the shearing stress.

1.b. In addition, a formula for the shear correction factor has also been de-
rived. The results obtained by the relationships set up for the stresses
are compared with finite element computations. A good agreement is
found between the different models.

As regards the corresponding publications see references (8), (12) and (19).
Though the title is the same for (12) and (19), the former one is more detailed.

Statement 2.
I have investigated the in-plane elastic static stability of circular beams with
cross-sectional inhomogeneity provided that the beam is subjected to a verti-
cal force at the crown point.

2.a. I have derived a new model both for the pre-buckling and post-buckling
radial displacements - in the later case both for symmetric and anti-
symmetric buckling. Cross-sectional inhomogeneity is implied in these
equations via the parameter m (which is a function of the E-weighted
radius of gyration and the radius of curvature). The equations I have
established are more accurate than those recently set up by Bradford et
al. in [4,24] for homogeneous and by Bateni and Eslami [5] for function-
ally graded material. Though I neglected the effect of the tangential
displacements on the angle of rotation, papers [4, 24] also apply this
assumption. Altogether, as the new model uses less neglects, the re-
sults for the critical load are more accurate than those published in the
formerly cited works.

2.b. Solutions are provided for (a) pinned-pinned, (b) fixed-fixed and (c)
rotationally restrained beams. For each case I have determined what
character the stability loss can have: no buckling, limit point buckling,
bifurcation buckling after limit point buckling, bifurcation buckling pre-
cedes limit point buckling. The endpoints of the corresponding intervals
are not constant in the modified slenderness λ as in the previous models
but further depend on the parameter m (on the E-weighted radius of
gyration and the radius of curvature).

2.c. Comparisons have been made with previous results and finite element
computations as well. These prove that the results obtained are applica-
ble also for not strictly shallow beams, up until the semi-vertex angle ϑ
is not greater than 1.5. For small central angles the differences between
the models are, in general, smaller than for greater central angles.

2.d. Cross-sectional inhomogeneity can have a serious effect on the critical
load. This is proven via a simple example.

As regards the corresponding publications see references (2), (3), (5), (10), (11), (13)-(18) and (20).

STATEMENT 3.
I have investigated the vibrations of circular beams with cross-sectional inhomogeneity, subjected to a vertical force at the crown point.

3.a.   I have derived the governing equations of those boundary value problems which make it possible to determine how a radial load affects the natural frequencies. For pinned-pinned and fixed-fixed beams I have determined the Green function matrices assuming that the beam is prestressed by a central load. When computing these matrices I had to take into account that the system of ordinary differential equations that govern the problem is degenerated.

3.b.   Making use of the Green function matrices, I have reduced the self-adjoint eigenvalue problems set up for the eigenfrequencies to eigenvalue problems governed by homogeneous Fredholm integral equation systems – four homogenous Fredholm integral equation systems have been established. These integral equations can directly be used for those dead loads, which result in a constant, otherwise either negative or positive axial strain on the E-weighted centerline. I have replaced these eigenvalue problems with algebraic ones and solved them numerically.

3.c.   It has turned out that the square of the quotient of the second loaded and unloaded natural frequencies depends almost linearly on the axial strain-critical strain ratio and is actually independent of the curved beam geometry and material inhomogeneity for pinned-pinned beams. The relations for fixed-fixed beams are more dependent on the central angle and are rather quadratic. In the knowledge of the load-strain relationship we can determine the strain due to the load, and then the natural frequencies of the loaded structure. If the strain is zero, we get back those results which are valid for the free vibrations.

3.d.   In some cases, the numerical results are verified by commercial finite element calculations and experiments as well. According to these, it turns out that the numerical model approximates the eigenfrequencies with a good accuracy.

As regards the corresponding publications see references (1), (4), (6), (7), (9), (11) and (20).

5   Possible application of the results

The results achieved can be applied to homogeneous or heterogeneous circular beams as structural elements to predict the behaviour (possible failure regarding the stresses, stability and vibrations) of the members under given circumstances. With new and improved models continuously being made, it is possible to gain more and more accurate results and thus, reduce uncertainties and save costs.

Some of the results could be harnessed in the education as nowadays non-homogeneous materials are gradually gathering ground. Primarily, I am thinking about the simple closed-form solutions for the normal and shear stress distributions in circular beams with cross-sectional inhomogeneity. Moreover, a simplified form of the stability model could as well be included in the curriculum to broaden the student's view of the phenomenon of buckling, which is many times restricted to the classical Euler column.

Moreover, the models and solutions obtained could be used for benchmark purposes to verify other models.

6   Future research

Based on the presented models, several additional improvements and generalizations could be made. In the simplest way, by changing the loading and/or the supports – even considering not symmetric conditions, or three-hinged beams – so that the investigations could be extended even more. Research is in progress for the vibration model when the beam is pinned at the left end and is fixed at the right end and there are equal rotational end restraints.

Keeping the hypotheses of the presented stability model, an interesting question is how the buckling loads, buckling shapes and the typical ranges change if the beam is subjected to a radial or vertical load at a point, other than the crown point. The post-buckling behaviour might also be worthy of dealing with and moreover, the dynamic behaviour could also be modeled some way. It would also be desirable to develop a one-dimensional finite element model, taking finite strains and/or rotations into account when dealing with the stability problem.

But such questions could as well be arisen how to harness the experiences of the presented models to tackle some issues of curved but not circular beams, out-of plane problems, bi-modular materials, to account for shear deformations, interlayer slip, etc.

It would also be satisfying to verify the results with experiments. Concerning this idea, there is an ongoing cooperation with some generous colleagues of the Transilvania University of Braşov.

7   The candidate's relevant publications

Journal articles in foreign language

(1) L. KISS AND GY. SZEIDL: Vibrations of pinned-pinned heterogeneous circular beams subjected to a radial force at the crown point. *Mechanics Based Design of Structures and Machines: An International Journal*, **43**(4), 2015, 424-449.

(2) L. KISS AND GY. SZEIDL: Nonlinear in-plane stability of heterogeneous curved beams under a concentrated radial load at the crown point. *Technische Mechanik*, **35**(1), 2015, 1-30.

(3) L. KISS AND GY. SZEIDL: In-plane stability of fixed-fixed heterogeneous curved beams under a concentrated radial load at the crown point. *Technische Mechanik*, **35**(1), 2015, 31-48.

(4) L. KISS, GY. SZEIDL, S. VLASE, B. P. GÁLFI, P. DANI, I. R. MUNTEANU, R. D. IONESCU AND J. SZÁVA: Vibrations of fixed-fixed heterogeneous curved beams loaded by a central force at the crown point. *International Journal for Engineering Modelling*, **27**(3-4), 2014, 85-100.

(5) L. KISS: In-plane buckling of rotationally restrained heterogeneous shallow arches subjected to a concentrated force at the crown point. *Journal of Computational and Applied Mechanics*, **9**(2), 2014, 171-199.

Journal articles in Hungarian language

(6) KISS L. P.: Heterogén anyagú síkgörbe rúd szabadrezgéseinek saját-frekvenciái, *GÉP*, **LXIV**(5), (2013), 16-21.

(7) KISS L. P. AND SZEIDL GY.: Tetőpontjában sugárirányú koncentrált erővel terhelt heterogén anyagú síkgörbe rúd rezgései, *Multidiszciplináris tudományok: A Miskolci Egyetem közleménye*, **3**(1-2), (2013), 67-82.

(8) KISS L. P.: Heterogén síkgörbe rudak lehetséges mechanikai modellje, *Multidiszciplináris tudományok: A Miskolci Egyetem közleménye*, **2**(1), (2012), 61-76.

Conference papers in book

(9) GY. SZEIDL AND L. KISS (Editor: S. Vlase): *Vibrations of heterogeneous curved beams subjected to a radial force at the crown point*, Proceedings of the 5th International Conference Computational Mechanics and Virtual Engineering, COMEC 2013, 24 - 25 October 2013, Braşov, Romania, pp. 24-33. ISBN: 978-606-19-0225-5.

(10) Gy. Szeidl and L. Kiss (Editor: S. Vlase): *A nonlinear mechanical model for heterogeneous curved beams*, Proceedings of the 4th International Conference on Advanced Composite Materials Engineering, CO-MAT, 18 - 20 October 2012, Braşov, Romania, Volume 2, pp. 589-596. ISBN 0981730051.

(11) Gy. Szeidl and L. Kiss (Editor: S. Vlase): *Vibrations and stability of heterogeneous curved beams*, Proceedings of the 4th International Conference on Computational Mechanics and Virtual Engineering COMEC 2011, 20 - 22 October 2011, Braşov, Romania, pp. 471-476. ISBN 978-973-131-122-7.

(12) L. Kiss And Gy. Szeidl: *Stresses in curved beams made of heterogeneous materials*, microCAD 2011: International Scientific Conference, 31 March - 1 April 2011, Miskolc, Hungary, Section: Applied Mechanics, pp. 13-18. ISBN 978-963-661-958-9.

Conference papers on CD

(13) Gy. Szeidl and L. Kiss: *Stability analysis of pinned-pinned shallow circular beams under a central concentrated load.* microCAD 2014: International Multidisciplinary Scientific Conference, 10 - 11 April 2014, Miskolc, Hungary, Section D4: Mechanical Modelling and Finite Element Simulation, Paper 40., 8p. ISBN 978-963-358-051-6.

(14) L. Kiss: *Stability of heterogeneous curved beams: A nonlinear formulation of the problem.* microCAD 2013: International Scientific Conference, 21 - 22 March 2013, Miskolc, Hungary, Section: Applied Mechanics, Paper 7., 6p. ISBN 978-963-358-018-9.

(15) L. Kiss: *In-plane stability of heterogeneous circular arches*, 8th International Conference of PhD Students, 6 - 10 August 2012, Miskolc, Hungary, Section: Engineering Sciences, Paper 9., 8p. ISBN 978-963-661-994-7.

(16) Gy. Szeidl And L. Kiss: *Stability of heterogeneous shallow arches subjected to a concentrated dead load*, microCAD 2012: International Scientific Conference, 29 - 30 March 2013, Miskolc, Hungary, Paper 9., 8p. ISBN 978-963-661-773-8.

Conference papers in Hungarian language

(17) Kiss L. and Szeidl Gy.: *Heterogén lapos görbe rudak stabilitásvizsgálata*, OGÉT 2012, 20th International Conference on Mechanical Engineering, 19 - 22 April 2012, Cluj-Napoca, Romania, pp. 234-237. ISSN 2068-1267.

Full papers that appeared in other[1] proceedings

(18) KISS L.: Heterogén anyagú lapos görbe rudak stabilitásvizsgálata. *Diáktudomány: A Miskolci Egyetem Tudományos Diákköri Munkáiból 2011-2012.* (2012), pp. 82-88. ISSN 2062-07-21.

(19) KISS L.: Stresses in Curved Beams Made of Heterogeneous Materials. *Diáktudomány: A Miskolci Egyetem Tudományos Diákköri Munkáiból 2010-2011.* (2011), pp. 51-56. ISSN 2062-07-21.

Conference talks

(20) KISS L., SZEIDL GY.: Heterogén anyagú síkgörbe rudak szabadrezgéseinek és stabilitásának vizsgálata. 9th Hungarian Conference on Theoretical and Applied Mechanics. 29-31 August 2011, Miskolc.

References

[1] I. Kozák and Gy. Szeidl. *Chapters from the Strength of Materials.* University of Miskolc, 2012. (in Hungarian).

[2] B. Csizmadia and E. Nándori. *Engineering Mechanics: Strength of Materials.* Nemzeti Tankönyvkiadó, 2002. (in Hungarian).

[3] I. Ecsedi and K. Dluhi. A linear model for the static and dynamic analysis of non-homogeneous curved beams. *Applied Mathematical Modelling*, 29:1211–1231, 2005.

[4] M. A. Bradford, B. Uy, and Y.-L. Pi. In-plane elastic stability of arches under a central concentrated load. *Journal of Engineering Mechanics*, 128(7):710–719, 2002.

[5] M. Bateni and M. R. Eslami. Non-linear in-plane stability analysis of FGM circular shallow arches under central concentrated force. *International Journal of Non-Linear Mechanics*, 60:58–69, 2014.

[6] Gy. Szeidl. *Effect of Change in Length on the Natural Frequencies and Stability of Circular Beams.* PhD thesis, Department of Mechanics, University of Miskolc, Hungary, 1975. (in Hungarian).

[7] E. Tüfekçi and A. Arpaci. Exact solution of in-plane vibrations of circular arches with account taken of axial extension, transverse shear and rotatory inertia affects. *Journal of Sound and Vibration*, 209(5):845–856, 1997.

[8] B. Kovács. Vibration analysis of layered curved arch. *Journal of Sound and Vibration*, 332:4223–4240, 2013.

[9] F. P. Beer and E. R. Johnston. *Mechanics of Materials.* Mc Graw Hill, Metric edition, 1987.

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[1] not regularly published

[10] L. Ascione and F. Fraternali. A penalty model for the analysis of curved composite beams. *Computers & Structures*, 45(5/6):985–999, 1991.

[11] J. M. Segura and G. Armengaud. Analytical formulation of stresses in curved composite beams. *Archive of Applied Mechanics*, 68:206–213, 1998.

[12] A. Baksa and I. Ecsedi. A note on the pure bending of nonhomogenous prismatic bars. *International Journal of Mechanical Engineering Education*, 37(2):118–129, 2009.

[13] E. Hurlbrink. Berechnung von rohrenartigen Kärpern, die unter ausserem Drucke stehen. *Schiffbau*, 9(14):517–523, 1907-1908.

[14] E. Chwalla and C. F. Kollbrunner. Beiträge zum Knickproblen des Bogantrâgers und des Rahmens. *Sthalbau*, 11(10):73–78, May 1938.

[15] D. A. DaDeppo and R. Schmidt. Sidesway buckling of deep crcular arches under a concentrated load. *Journal of Applied Mechanics,ASME*, 36(6):325–327, June 1969.

[16] Y.-L. Pi and N. S. Trahair. Non-linear buckling and postbuckling of elastic arches. *Engineering Structures*, 20(7):571–579, 1998.

[17] E. Volterra and J. D. Morrel. Lowest natural frequency of elastic arc for vibrations outside the plane of initial curvature. *Journal of Applied Mechanics*, 12:624–627, 1961.

[18] S. Timoshenko. *Vibration problems in engineering*. D. Van Nonstrand, 1955.

[19] K. Kang, C. W. Bert, and A. G. Striz. Vibration analysis of shear deformable circular arches by the differential quadrature method. *Journal of Sound and Vibration*, 181(2):353–360, 1995.

[20] G. Szeidl, K. Kelemen, and Á. Szeidl. Natural frequencies of a circular arch – computations by the use of Green functions. *Publications of the University of Miskolc, Series D. Natural Sciences, Mathematics*, 38:117–132, 1998.

[21] K. Kelemen. Vibrations of circular arches subjected to hydrostatic follower loads – computations by the use of the Green functions. *Journal of Computational and Applied Mechanics*, 1(2):167–178, 2000.

[22] X. Y. Li, X. Zhao, and Y. H. Li. Green's functions of the forced vibration of Timoshenko beams with damping effect. *Journal of Sound and Vibration*, 333:1781–1795, 2014.

[23] Y. L. Pi and M. A. Bradford. Non-linear in-plane analysis and buckling of pinned-fixed shallow arches subjected to a central concentrated load. *International Journal of Non-Linear Mechanics*, 47:118–131, 2012.

[24] Y. L. Pi, M. A. Bradford, and F. Tin-Loi. Non-linear in-plane buckling of rotationally restrained shallow arches under a central concentrated load. *International Journal of Non-Linear Mechanics*, 43:1–17, 2008.