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Asymptotic characterisation of dynamic linear systems with small time delay

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Asymptotic characterization of dynamic linear systems with small time delay

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Written by Áron Fehér

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'If you love God, you can't hate anything or anyone. If the love one offers is met with hate, it doesn't die, rather it manifests in the form of compassion. That is universal love. It is not just a sentiment. It cannot be manifested merely by a shift in mental disposition. It can only come from inner cleaning, an inner awakening.'

Radhanath Swami

Abstract

Accurate modelling, analysis and control techniques are essential to ensure the proper operation of modern processes. System and control theory provide well-established methods for the analysis and control problems of linear time-invariant systems. The complexity of some physical and bio-chemical phenomena yields to time delay into the dynamics (e.g. communication delay in robotic swarms and vehicle platoons, data processing delays in distributed algorithms, reaction times of chemical reaction networks, delay caused by intracellular molecular motions in biological systems). The time delay cannot be neglected in most cases, so it is necessary to introduce new analysis and synthesis methods for such systems.

This dissertation aims to provide a refined approximation method for both continuous- and discrete-time linear time-delay systems and to apply the technique in analysis and control scenarios.

It is shown that if a certain *smallness condition* holds, then the time delay system can be approximated exponentially fast with a delay-free system of ordinary differential equations. The state variable of the approximating system has the same dimensions as the state variable of the original system. The state matrix of the approximating system is given as the solution of an exponential matrix equation. The eigenvalues of the approximating system coincide with the dominant eigenvalues of the original system.

An exponentially convergent iterative algorithm is given to compute the state matrix from the analytical solution based on Banach's fixed point theorem, with error metric for comparison with the analytical solution.

The homogeneous time-delay system is extended with constant non-homogeneous term, and both analytical and iterative approaches are given to find the approximating non-homogeneous system without delay.

The developed approximation method is discussed and applied within the framework of three system classes.

In the case of continuous-time linear systems with point-wise delay, the method was used to study detectability and to design a classical observer system.

The discrete-time version of the approximation method was developed for the approximation of Volterra-type difference systems containing infinite delays. The approximation method was applied to approximate and analyse multi-agent systems with communication delay.

Furthermore, an approximation method for continuous-time linear systems with distributed delay has been developed, and it was applied for system analysis and control design.

In all three cases, simulation results show the applicability of the proposed analysis and synthesis methods.

Kivonat

A korszerű folyamatok tervszerű működésének biztosítása érdekében elengedhetetlen a folyamatok pontos elemzése, modellezése és szabályozása. A rendszerelmélet és irányítástechnika jól bevált módszereket biztosít a lineáris időinvariáns rendszerek elemzésére és szabályozó szintézisére. A folyamatok bonyolultsága késleltetést hozhat a rendszerbe (robot rajokban vagy konvojokban fellépő kommunikációs késés, elosztott algoritmusokban jelenlevő adatfeldolgozási késés, kémiai reakcióhálók reakcióideje vagy biológiai rendszerek esetén sejten belüli molekuláris mozgás által okozott késleltetések). A késleltetést ezen folyamatok modellezése során nem tudjuk elhanyagolni, ezért szükség van új elemzési és szintézis módszerek bevezetésére.

A dolgozatban egy olyan módszer kerül bemutatásra, melynek segítségével egy késleltetett homogén differenciálegyenlet-rendszer megközelíthető egy közönséges homogén differenciálegyenlet-rendszerrel, ha egy bizonyos kicsinységi feltétel teljesül. A közelítő egyenletrendszer állapot változóinak száma megegyezik az eredeti késleltetett egyenletrendszer állapotváltozóinak számával. A közelítő egyenletrendszer sajátértékei megegyeznek az eredeti késleltetett egyenletrendszer domináns sajátértékeivel. A közelítő módszer konvergenciája exponenciális.

A dolgozatban analitikus egyenletet nyújtottunk a közelítő rendszer állapotmátrixának meghatározására. Ezen exponenciális mátrixegyenlet megoldásának könnyítését egy iteratív módszer teszi lehetővé a Banach-féle fixpont tétel alkalmazásával. Továbbá tárgyalva van a késleltetett egyenletrendszerben esetlegesen szereplő nem-homogén tag átvitele a közelítő rendszerbe, amire egy analitikus módszer és numerikus megközelítés van bemutatva.

A kidolgozott közelítő módszer és alkalmazásai három rendszerosztály keretén belül vannak tárgyalva.

Pontszerű késleltetést tartalmazó folytonos idejű lineáris időinvariáns rendszerek esetén az említett módszer a detektálhatóság vizsgálatára és megfigyelőrendszer tervezésére volt alkalmazva.

A közelítő módszer diszkrét változata végtelen késleltetést tartalmazó Volterra-féle differenciaegyenlet-rendszerekre lett kidolgozva, melyet a szerző kommunikációs késleltetéssel rendelkező multi-ágens rendszerek approximációjára alkalmazott.

Továbbá ki lett dolgozva a közelítő módszer elosztott késleltetést tartalmazó folytonos idejű lineáris időinvariáns rendszerekre, melynek keretén belül rendszerelemzésre, stabilizálhatóság vizsgálatra és szabályozó tervezésre lett alkalmazva.

Mindhárom esetben a javasolt módszerek alkalmazhatóságát a szerző szimulációkkal támasztotta alá.

Rezumat

Metodele de modelare, analiza și control a sistemelor dinamice sunt esențiale pentru a asigura funcționarea planificată a proceselor moderne. Teoria sistemelor și teoria reglării automate oferă metode bine stabilite pentru analiza și sinteza regulatoarelor pentru sisteme liniare. Complexitatea proceselor poate aduce întârzieri în sistem (întârzieri de comunicare în roiurile de roboți, întârzieri de procesare a datelor în algoritmi distribuiți, timpi de reacție ai rețelelor de reacții chimice sau întârzieri cauzate de mișcarea moleculară intracelulară în sistemele biologice). Timpul mort nu pot fi neglijat în majoritatea cazurilor, așa că este necesar dezvoltarea unor metode speciale pentru sisteme cu timp mort.

Disertația prezintă o metodă de approximare pentru aceste sisteme prin care modelul unui sistem dinamic cu întârzieri poate fi aproximat cu un sistem de ecuații diferențiale fără întârzieri dacă este îndeplinită o condiție specială legată de întârzieri. Dimensiunea vectorului de stare al sistemul de aproximare este egal cu dimensiunea vectorului de stare al sistemul inițial cu întârzieri. Valorile proprii ale sistemului de ecuații aproximative coincid cu valorile proprii dominante ale sistemului original.

Disertația oferă o ecuație analitică pentru determinarea matricei de stare al sistemului de approximare de ecuații. Soluția acestei ecuații matriciale exponențiale se poate obține folosind o metodă numerică utilizând teorema punctului fix al lui Banach. Convergența metodei de aproximare numerice este exponențială. Mai mult, este discutat introducerea unui termen neomogen în sistemul de ecuații cu întârzieri, pentru care este prezentat o metodă analitică și una numerică pentru a determina termenul neomogen pentru sistemul de aproximare.

Metoda de aproximare dezvoltată este discutată pentru trei clase de sisteme.

În cazul sistemelor liniare invariante în timp continuu cu întârziere, metoda menționată a fost utilizată pentru a studia detectabilitatea sistemului și pentru a proiecta un estimator de stare.

O versiune discretă a metodei de aproximare a fost dezvoltată pentru sistemele Volterra de ecuații de diferență cu întârziere infinită, pe care autorul le-a folosit pentru a analiza comportamentul dinamic al sistemelor tip multi-agent cu întârziere de comunicare.

Mai mult, a fost dezvoltată o metodă de aproximare pentru sisteme liniare în timp continuu cu întârziere distribuită, și a fost aplicat pentru analiza controlabilității și proiectarea regulatoarelor tip reacție de stare.

În toate cele trei cazuri, aplicabilitatea metodelor propuse a fost demonstrată cu simulări.

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List of Abbreviations

DDE	**D**elay **D**ifferential **E**quation
DIDE	**D**elay **I**ntegro-**D**ifferential **E**quation
LTI	**L**inear **T**ime **I**nvariant
MAS	**M**ulti **A**gent **S**ystem
ODE	**O**rdinary **D**ifferential **E**quation
TDS	**T**ime **D**elay **S**ystem

List of Symbols

$\mathbb{N}$	the set of all natural numbers
$\mathbb{N}^*$	the set of non-zero natural numbers, $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$
$\mathbb{Z}$	the set of all integer numbers
$\mathbb{Z}_+$	the set of non-negative integer numbers
$\mathbb{Z}_-$	the set of non-positive integer numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^*$	the set of non-zero real numbers, $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$
$\mathbb{R}_+$	the set of non-negative real numbers, $\mathbb{R}_+ = [0, \infty)$
$\mathbb{C}$	the set of complex numbers
$\mathbb{C}^*$	the set of non-zero complex numbers, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$
$\mathbb{R}^n$	the set of column vectors with $n \in \mathbb{N}^*$ real elements
$\mathbb{C}^n$	the set of column vectors with $n \in \mathbb{N}^*$ complex elements
$\mathbb{R}^{n\times m}$	the set of $n \times m$ matrices with real elements, where $m, n \in \mathbb{N}^*$
$\mathbb{C}^{n\times m}$	the set of $n \times m$ matrices with complex elements, where $m, n \in \mathbb{N}^*$
$\lvert\mathcal{A}\rvert$	the cardinality of the set $\mathcal{A}$, i.e. the number of elements in $\mathcal{A}$
$\Re(z)$	real part of $z \in \mathbb{C}$
$\Im(z)$	imaginary part of $z \in \mathbb{C}$
$\lvert x\rvert$	absolute value of a scalar x
$\underline{x} \in \mathbb{R}^n$	a column vector in $\mathbb{R}^n$ with elements, $\underline{x} = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}^\top$
$\lVert\underline{x}\rVert_p$	p-norm of $\underline{x} \in \mathbb{R}^n$ defined as $\lVert\underline{x}\rVert_p = \left(\sum_{i=1}^n \lvert x_i\rvert^p \right)^{1/p}$
$\lVert\underline{x}\rVert_\infty$	infinity norm of $\underline{x} \in \mathbb{R}^n$ defined as $\lVert\underline{x}\rVert_\infty = \max_{1\le i\le n} \lvert x_i\rvert$
$\underline{1}_n$	all-ones vector in $\mathbb{R}^n$
$\underline{0}_n$	zero vector in $\mathbb{R}^n$
$A \in \mathbb{R}^{m\times n}$	an $m \times n$ matrix with real elements
$A^\top \in \mathbb{R}^{n\times m}$	transpose of A
A^{-1}	inverse of $A \in \mathbb{R}^{n\times n}$
A^{-j}	power of the inverse matrix, $A^{-j} = (A^{-1})^j$ for $j \in \mathbb{N}$
$A^\dagger \in \mathbb{R}^{n\times m}$	Moore-Penrose inverse of A
e^A	matrix exponential of a square matrix A defined as $\mathrm{e}^A = \sum_{k=0}^\infty (1/k!)A^k$
$\lVert A\rVert$	matrix norm induced by the given vector norm $\lVert\underline{x}\rVert$
$\mathrm{rank}(A)$	the rank of A, i.e. the dimension of the vector space generated by its columns
I_n	$n \times n$ identity matrix
O_n	$n \times n$ zero matrix
$\mathrm{diag}(\underline{x})$	a square matrix with entries d_{ij}, where $d_{ij} = x_i$ if $i = j$, otherwise $d_{ij} = 0$.
$\det(A)$	determinant of matrix A
$\sigma(A)$	spectrum of a square marix A i.e. the set of eigenvalues

$\rho(A)$	spectral radius of a square matrix A defined as $\rho(A) = \sup_{\lambda\in\sigma(A)} \lvert\lambda\rvert$
$\mathcal{C}(\mathbb{R}, \mathbb{R}^n)$	the space of continuous functions $\underline{f} : \mathbb{R} \to \mathbb{R}^n$
$\lVert\underline{f}\rVert_{\mathcal{C}}$	supremum norm of $\underline{f} \in \mathcal{C} := \{\underline{f} \mid \underline{f} : A \to B\}$ defined as $\lVert\underline{f}\rVert_{\mathcal{C}} = \sup_{t\in A} \lVert\underline{f}(t)\rVert$
$\underline{x}(t)$, or $\underline{x}[n]$	state of a given system
$\underline{y}(t)$, or $\underline{y}[n]$	output of a given system
$\underline{u}(t)$, or $\underline{u}[n]$	input of a given system
$\dot{\underline{x}}(t) = \frac{\mathrm{d}\underline{x}}{\mathrm{d}t}$	continuous time derivative
$\Delta\underline{x}[n] = \frac{\underline{x}[n]-\underline{x}[n-1]}{T_s}$	discrete time forward difference operator with sampling time T_s

Throughout this thesis, my own papers are cited as $[j*]$, while other publications are cited as $[j]$.

Chapter 1

Introduction

The analysis and control problem of general dynamic systems that contain time delays is an interesting subject mainly because of the infinite-dimensional property of the dynamic system models with delay. In most cases, the classical system- and control theoretical methods cannot be successfully applied to such systems. New, computational heavy algorithms are necessary for the analysis and the design of delay systems. However, in some special cases, a time-delay system can be uniquely approximated with ordinary differential equations, and in such cases, the analysis and control design methods, developed for delay-free systems, can still be applied.

The present work deals with the approximation of three different delay system classes with different applications, all connected through system- and control theory. The first one is the approximation of a class of continuous-time linear systems with point-wise delay and its application to observer design. The second one is the approximation of discrete-time Volterra-type difference systems containing infinite delays with multi-agent systems application. The final one treats the approximation of systems with distributed delays and its application to controller design.

1.1 Background and motivation

TDSs, also known as systems with dead-time, differential-difference systems, or hereditary systems, are a class of functional differential systems. In contrast with ODEs, the TDSs are infinite-dimensional, they can usually be solved with the method of steps, and the solutions are not always backwards continuable [1]. The following points could explain the importance of these system classes:

- Expectation of models with better performance and close resemblance to real systems. The majority of dynamic systems in biology, mechanics, physiology, chemistry and economics include internal delays [2]. The delay also has a crucial effect on the stability of networked control systems [3] or high-speed communication networks [4].
- The classic control design methods cannot be applied to TDSs in most cases, in a sense that ignoring the delay or simply replacing the DDE with ODE results in different behavior of the approximate model [5].
- The delay term can have stabilising, destabilising effects or it can induce chaotic behaviour. In some cases, the introduction of time delay in the feedback loop of an ODE system dampens the output [6]. In contrast, a sufficiently large time delay creates limit cycles and it can induce chaotic behaviour [7].
- TDSs can sometimes simplify system models with high degree [8] or systems with partial differential (transport) equations [9].

Based on the above mentioned points of interests, the motivation of my work was to develop a computationally simple approximation method for delay systems with small delay. Furthermore, the goal was to use this approximation to extend the classical system- and control theoretical methods such as state estimation and state feedback control to the addressed class of systems. In particular:

- to show that under specific condition there exists an ODE which is asymptotically equivalent to the original TDS.
- to give explicit equations to calculate the state matrices and nonhomogeneous terms of the approximate system based on the original TDS.
- to give iterative methods which can be used to approximate the solution of the above mentioned explicit equations.
- to formulate simplified detectability and stabilisability conditions for the TDS based on their approximate ODEs.
- to synthesize full state observers and state feedback control laws based on the approximation method.

1.2 Theoretical background

This section provides a short theoretical background which is used as the backbone for the latter chapters. Both ODEs and DDE are discussed highlighting the similarities and main differences between them. The arosen difficulties are highlighted in the case of DDEs in the fields of engineering.

1.2.1 Linear ordinary differencial and difference equations

Continuous-time systems

ODEs are differential equations containing one or more functions of one independent variable and the derrivatives of those functions [10], in contrast with partial differential quations which may contain more than one independent variable.

A *linear* n^{th} order ODE is a differential equation of the form

$$\alpha_0(t)x + \alpha_1(t)x' + \cdots + \alpha_n(t)x^{(n)} = \beta(t), \tag{1.1}$$

with initial condition $x(0) = x_0$, $x'(0) = x'_0$, $\ldots$, $x^n(0) = x_0^n$, where $\alpha_0(t)$, $\alpha_1(t)$, $\ldots$, $\alpha_n(t)$, $\beta(t)$ are arbitrary continuous functions, x', $\ldots$, $x^{(n)}$ are the successive derivates of $x : \mathbb{R} \to \mathbb{R}$, which is a function of time t.

If $\beta(t) = 0$, the ODE is homogeneous, otherwise it is nonhomogeneous.

If $\alpha_0(t) = \alpha_0, \alpha_1(t) = \alpha_1, \ldots, \alpha_n(t) = \alpha_n$, the ODE is time independent, otherwise it is time dependent.

A function $z(t)$ is called a general solution of (1.1) on some interval $\mathcal{I}$, if it is n-times differentiable on $\mathcal{I}$ and it satisfies (1.1) for all $t \in \mathcal{I}$.

In engineering application, in most cases the independent variable is time, and the systems are modelled using first ordered co-dependent linear autonomous ODEs

$$\begin{cases} \dot{x}_1(t) = a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t) + b_1(t) \\ \dot{x}_2(t) = a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2n}x_n(t) + b_2(t) \\ \vdots \\ \dot{x}_n(t) = a_{n1}x_1(t) + a_{n2}x_2(t) + \cdots + a_{nn}x_n(t) + b_n(t) \end{cases}, \tag{1.2}$$

with initial condition $x_1(0) = x_{10}$, $x_2(0) = x_{20}$, $\dots x_n(0) = x_{n0}$, which can be writen in vectorial form as

$$\dot{\underline{x}}(t) = A\underline{x}(t) + \underline{b}(t), \quad \underline{x}(0) = \underline{x}_0, \tag{1.3}$$

with $\underline{x} = \begin{pmatrix} x_1 & x_2 & \dots & x_n \end{pmatrix}^\top$, $\underline{b} = \begin{pmatrix} b_1 & b_2 & \dots & b_n \end{pmatrix}^\top$ and $A = (a_{ij})$ for $1 \le i, j \le n$. The homogeneous part of (1.3) is

$$\dot{\underline{x}}(t) = A\underline{x}(t) \quad \underline{x}(0) = \underline{x}_0. \tag{1.4}$$

When solving the system of ODE (1.3) on some interval $\mathcal{I}$ with a given initial condition $\underline{x}(t_0) = \underline{x}_0$, the solution is always unique, the backward continuation is always possible, and it can be written as

$$\underline{x}(t) = \mathrm{e}^{At}\underline{x}_0 + \int_0^t \mathrm{e}^{A(t-s)}\underline{b}(s)\mathrm{d}s, \quad t \in \mathcal{I}. \tag{1.5}$$

The characteristic equation of (1.4) is

$$\det(\lambda I_n - A) = 0, \tag{1.6}$$

which is an algebraic equation of degree n, and its n roots $\lambda_1, \lambda_2, \dots \lambda_n \in \mathbb{C}$ (counting multiplicities) can be used for the stability analysis or for the formulation of the general solution of (1.3).

Discrete-time systems

The discrete equivalent of (1.3), is written in the form

$$\Delta\underline{x}[k] = (I_n - A_d)\underline{x}[k] + \underline{b}_d[k], \quad \underline{x}[0] = \underline{x}_0, \tag{1.7}$$

with $A_d \in \mathbb{R}^{n\times n}$, and it is called a system of linear ordinary difference equations, where Δ is the forward difference operator. Similarly to the continuous case, a solution from the initial condition is

$$\underline{x}[k] = A_d^k\underline{x}_0 + \sum_{i=0}^{k-1} A_d^{i-k+1}\underline{b}_d[i], \tag{1.8}$$

is unique.

The characteristic equation of the homogeneous part of (1.7) is

$$\det(zI_n - I_n - A_d) = 0, \tag{1.9}$$

which has n roots $z_1, z_2, \dots z_n \in \mathbb{C}$ counting multiplicities.

The homogeneous part of (1.7) is

$$\Delta\underline{x}[k] = (I_n - A_d)\underline{x}[k]. \tag{1.10}$$

Discretized systems

A continuous model can be discretized, i.e. the system (1.3) can be transformed into a discrete-time system having the form (1.7) with the substitution $A_d = \mathrm{e}^{AT_s}$ and $\underline{b}_d[k] = \int_0^{T_s} \mathrm{e}^{As}\mathrm{d}s\underline{b}[k]$ or using one of approximate discretization methods:

- Forward Euler method, where $\mathrm{e}^{AT_s} \approx I_n + AT_s$
- Backward Euler method, where $\mathrm{e}^{AT_s} \approx (I_n - AT_s)^{-1}$
- Bilinear transform, where $\mathrm{e}^{AT_s} \approx (I_n + AT_s/2)(I_n - AT_s/2)^{-1}$

where T_s is the sampling time [11].

1.2.2 The introduction of state delay

If state delays are introduced in the system (1.3) the resulting TDS can be written as

$$\dot{\underline{x}}(t) = A_0\underline{x}(t) + A_\tau\underline{x}(t-\tau) + \underline{b}(t), \quad \underline{x}(h) = \underline{\phi}(h) \text{ for } h \in [-\tau, 0], \tag{1.11}$$

where $0 < \tau < \infty$ is the time delay and $\phi : [-\tau, 0] \to \mathbb{R}^n$ is a continuous initial function. The homogeneous part of (1.11) is

$$\dot{\underline{x}}(t) = A_0\underline{x}(t) + A_\tau\underline{x}(t-\tau), \quad \underline{x}(h) = \underline{\phi}(h) \text{ for } h \in [-\tau, 0]. \tag{1.12}$$

The characteristic equation of the system is

$$\det(\lambda I_n - A_0 - A_\tau \mathrm{e}^{-\tau\lambda}) = 0. \tag{1.13}$$

It can be seen that, due to the exponential term $A_\tau \mathrm{e}^{-\tau\lambda}$, in the complex plain the characteristic equation, in general, has infinitely many roots, which increases the difficulty of system analysis.

Furthermore, a DDE requires an initial function on the interval $[-\tau, 0]$ for the solution, and not every solution is backwards continuable [1].

1.2.3 The introduction of input delay

In system theory, a linear time invariant dynamic system is modelled as a nonhomogeneous system of ODE (1.3), where the nonhomogeneous term $\underline{b}(t)$ is a linear combination of the input signal $\underline{u}(t) : \mathbb{R} \to \mathbb{R}^m$ with an input gain matrix $B \in \mathbb{R}^{n\times m}$ such that $\underline{b}(t) = B\underline{u}(t)$ [12].

If input delays are present in the model then the input the system (1.3) becomes

$$\dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t-\tau), \quad \underline{x}(h) = \underline{\phi}(h) \text{ for } h \in [-\tau, 0], \tag{1.14}$$

In order for the states to converge to a given constant value, a control law is implemented in the system by feeding back a part or the full state vector in the input [13]. A full state static feedback can be formulated as $\underline{u}(t) = K\underline{x}(t)$, with $K \in \mathbb{R}^{m\times n}$ gain matrix. Using the previous feedback control, the system (1.14) becomes

$$\dot{\underline{x}}(t) = A\underline{x}(t) + BK\underline{x}(t-\tau), \tag{1.15}$$

which is a homogeneous DDE.

1.2.4 Discrete time difference systems with state delay

Consider the system of homogeneous ordinary difference equations with delay

$$\Delta \underline{x}[k] = A_0 \underline{x}[k] + A_q \underline{x}[k-q], \quad \underline{x}[h] = \underline{\phi}[h] \text{ for } h = -q, -q+1, \ldots, 0, \tag{1.16}$$

where $q \in \mathbb{N}^*$ is the discrete delay. Similarly to the continuous case, an initial function $\underline{\phi}$ on the discrete interval $[-q, 0]$ is required for the solution. In contrast to the continuous case the characteristic equation

$$\det(zI_n - I_n - A_0 - A_\tau z^{-q}) = 0 \tag{1.17}$$

has finitely many solutions.

The system

$$\Delta \underline{x}[k] = \sum_{j=0}^{\infty} A[j] \underline{x}[k-j] \quad \underline{x}[h] = \underline{\phi}[h] \text{ for } h \in \mathbb{Z}_-, \tag{1.18}$$

is called a Volterra difference equation with infinite delays. Here $A : \mathbb{Z}_+ \to \mathbb{R}^{n \times n}$ is a matrix function such that $A[j] \neq O_n$ for some $j \in \mathbb{Z}_+$ and $\underline{\phi} : \mathbb{Z}_- \to \mathbb{R}^{n \times n}$ is an initial function.

The eigenvalues $z \in \mathbb{C}$ of (1.18) are the roots of the characteristic equation

$$\det D(z) = 0, \quad \text{with } D(z) = (z-1)I_n - \sum_{j=0}^{\infty} A[j] z^{-j}. \tag{1.19}$$

1.2.5 The introduction of distributed delay

The use of discrete delay impicitly assumes that the system dynamics is modelled with the use of δ-Dirac distribution, in other words, each individual time instance in the state variable is subjected to the same gain factor [14]. This technique may seem like a rough approximation in system modelling, and sometimes it would be more realistic to assume that the delay is distributed continuously by a continuous distribution funcion like

$$\dot{\underline{x}}(t) = A_0 \underline{x}(t) + \int_{-\tau}^{0} A_\tau(\eta) \underline{x}(t+\eta) \mathrm{d}\eta, \tag{1.20}$$

where $0 < \tau < \infty$, $A_0 \in \mathbb{R}^{n \times n}$ and $A_\tau : [-\tau, 0] \to \mathbb{R}^{n \times n}$ is a continuous delay distribution function, for which there exists $\eta \in [-\tau, 0]$, such that $A_\tau(\eta) \neq O_n$. The eigenvalues $\lambda \in \mathbb{C}$ of (1.20) are the roots of the characteristic equation

$$\det R(\lambda) = 0, \quad \text{where } R(\lambda) = \lambda I - A_0 - \int_{-\tau}^{0} A_\tau(\eta) \mathrm{e}^{\eta \lambda} \mathrm{d}\eta. \tag{1.21}$$

1.3 Approximation of TDS - survey for previous works

This section provides a review of the different approximation methods for delay systems based on [15*]. The form of the studied systems is shown by (1.12) which is a time-invariant, homogeneous TDS with constant initial condition.

There is a multitude of algorithms presented for the purpose of approximating TDSs. Some of these algorithms estimate only the roots of the characteristic equation (1.13) with a polynomial characteristic equation of specified degree (Taylor series- or

Padé-based approximations) or in a given region of interest (e.g. QPmR). Other methods approximate the trajectories of the delayed system with the trajectories of a well-constructed system of ODEs (e.g the modified chain approximation or Galerkin's method).

1.3.1 Algorithms for the approximation of eigenvalues

Approximation with Taylor series expansion

TDSs of given by the DDE (1.13) and their characteristic equations (1.13) are often approximated by ODEs and by their characteristic equation using Taylor series expansion in powers of τ [16].

This method involves higher-order derivatives of alternating signs, which could lead to an approximating system with different stability. The Taylor approximation of the characteristic equation (1.13) is written as

$$\det(\lambda I_n - A_0 - A_\tau(1 - \tau\lambda + \frac{1}{2}\tau^2\lambda^2 + \cdots + \frac{(-1)^p}{p!}\tau^p\lambda^p)) = 0.$$

It has been shown that this method only works in a few cases. In other cases, it may lead to qualitatively different systems [17].

Approximation with Padé series expansion

The Padé approximation of TDS is based on the Taylor series expansion and the Padé model reduction. A fractional-order Padé approximation method was developed based on optimal polynomial fitting as an approximation of $e^{-\tau\lambda}$ in (1.13) in a given region of interest [18].

The characteristic equation (1.13) can be written as

$$\det(\lambda I_n - A_0 - A_\tau \frac{1 - \frac{\tau\lambda}{2} + \frac{\tau^2\lambda^2}{9} - \frac{\tau^3\lambda^3}{72} + \frac{\tau^4\lambda^4}{1008} - \frac{\tau^5\lambda^5}{30240}}{1 + \frac{\tau\lambda}{2} - \frac{\tau^2\lambda^2}{9} + \frac{\tau^3\lambda^3}{72} - \frac{\tau^4\lambda^4}{1008} + \frac{\tau^5\lambda^5}{30240}}) = 0.$$

using 5^{th} order Padé series expansion

Approximation with the Lambert W function

The Lambert W function is the multivalued inverse of the function $E : \mathbb{C} \to \mathbb{C}$ defined by $E(z) = ze^z$ for $z \in \mathbb{C}$ [19].

In the scalar case the rightmost eigenvalues of (1.12) can be computed by numerical evaluations of the branches of the Lambert W function. The method can be extended to systems [20, 21].

Approximation using quasi-polynomial root finder algorithm

The quasi-polynomial root finder algorithm calculates the rightmost eigenvalues of a TDS based on the characteristic equation (1.13), which can be expressed as

$$P(\lambda) = \sum_{k=0}^{N} Q_k(\lambda) e^{-\alpha_k\tau\lambda},$$

where Q_k is a polynomial expression with real coefficients, and $\alpha_k \in \mathbb{R}$. The objective is to locate the eigenvalues in the complex plane region $\mathbb{D} \subset \mathbb{C}$ with boundaries

$\beta_{min} < \Re(\mathbb{D}) < \beta_{max}$ and $\omega_{min} < \Im(\mathbb{D}) < \omega_{max}$ as intersection points of the zero level curves of the surfaces $\Re(P(\beta + j\omega)) = 0$, $\Im(P(\beta + j\omega)) = 0$ [20, 22].

1.3.2 Approximation methods for the solutions of the DDE

Approximation using the modified chain method

The modified chain approximation method introduced by Repin [23] builds an approximating system directly from the state space representation of the TDS. The approximating system for a TDS described by (1.12) is written as

$$\begin{cases} \dot{\underline{\hat{x}}}_0(t) = A_0\underline{\hat{x}}_0(t) + \frac{m}{\tau}\underline{\hat{x}}_m(t) \\ \dot{\underline{\hat{x}}}_1(t) = A_\tau\underline{\hat{x}}_0(t) - \frac{m}{\tau}\underline{\hat{x}}_1(t) \\ \vdots \\ \dot{\underline{\hat{x}}}_k(t) = \frac{m}{\tau}\underline{\hat{x}}_{k-1}(t) - \frac{m}{\tau}\underline{\hat{x}}_k(t) \end{cases}, \quad 2 \leq k \leq m, \tag{1.22}$$

where $\hat{x}_p \in \mathbb{R}^n$ for $p = 0, 1, \ldots, m$, with initial condition $\underline{\hat{x}}_0(0) = \underline{x}_0$ and $\underline{\hat{x}}_p(0) = (\tau/m)\underline{x}_0$, where $1 \leq p \leq m$, and output $\underline{y} = \underline{x}_0$ The state $\underline{\hat{x}}_0$ represent the approximation for the states of (1.12), and this is a linearly convergent approximation [24].

Approximation using Galerkin's method with tau incorporation

In numerical analysis, Galerkin's method is used to convert a continuous operator problem in a weak formulation to a discrete problem by applying constraints determined by a finite set of basis functions [25].

Define the state transformation $\underline{x}(t+s) = \Phi^\top(s)\underline{\eta}_i(t)$, where $s \in [-\tau, 0]$, and $t \in \mathbb{R}_+$. Here $\Phi(s)^\top \in \mathbb{R}^{np\times n}$ is a vector containing a finite number of basis functions, while $\underline{\eta}(t) \in \mathbb{R}^{np\times 1}$ are the time dependent coordinates, $\mathbb{N} \ni p \geq 2$ is the degree of approximation.

The approximating equation can be writen in the descriptor form as

$$\begin{pmatrix} \Gamma \\ \Phi^\top(0) \end{pmatrix} \dot{\underline{\eta}}(t) = \begin{pmatrix} \Psi \\ A_0\Phi^\top(0) + A_\tau\Phi^\top(-\tau) \end{pmatrix} \underline{\eta}(t), \tag{1.23}$$

where

$$\Gamma = \int_{-\tau}^{0} \Phi(s)\Phi^\top(s)\mathrm{d}s,$$

$$\Psi = \int_{-\tau}^{0} \Phi(s)\frac{\mathrm{d}}{\mathrm{d}s}\Phi^\top(s)\mathrm{d}s,$$

the initial condition is given by

$$\underline{\eta}(0) = \Gamma^{-1}\int_{-\tau}^{0} \Phi(s)\mathrm{d}s\underline{x}_0,$$

and the approximating states are $\underline{\hat{x}}(t) = \Phi^\top(0)\underline{\eta}(t)$.

The system (1.23) is overdetermined, and the solution involves the application of least-squares fitting [26]. The tau incorporation or spectral tau method creates the descriptor system

$$\overline{\Gamma}\dot{\underline{\eta}}(t) = \overline{\Psi}\underline{\eta}(t),$$

such that $\overline{\Gamma} := \Gamma$, and every in^{th} row is replaced by the i^{th} row of $\Phi^\top(0)$. Similarly, $\overline{\Psi} := \Psi$, and every in^{th} row is replaced by the i^{th} row of $A_0\Phi^\top(0) + A_\tau\Phi^\top(-\tau)$, where $i = 1, 2, \ldots, n$. This altered form is not over-constrained, so there is no need for a fitting algorithm, and the approximation characteristics are improved [27].

The details of this approximation method are shown in Appendix A.

Approximation using spectral and pseudospectral methods

Breda et al [28] proposed the pseudospectral collocation method. Similarly to the previously shown Galerkin's method, this procedure approximates the TDS using the method of weighted residuals with Lagrange base polynomials and the boundary conditions are enforced with the tau incorporation. Butcher and Bobrenkov [29] extended this method to linear and nonlinear systems of TDS with time-periodic coefficients.

Lehotzky [30] proposed two numerical methods for the finite dimensional approximation of TDS the pseudospectral tau and the spectral element methods. The former is a weighted residual type method, similar to Galerkin's approximation using Lagrange base polynomials, where the analytical integration is substituted by the numerically feasible Lobatto-type Legendre-Gauss quadratures [31]. The latter approximates the TDS operator equation using weighted quadrature nodes together with the Lobatto-type Legendre-Gauss quadratures (or Clenshaw-Curtis quadrature [32] for increased accuracy). Furthermore, the author compared the proposed methods with Galerkin's approximation and the Pseudospectral collocation methods. It was stated that the efficiency of the algorithm could be improved if only the critical eigenvalues are calculated [33]. Moreover, the efficiency can be increased by using non-uniform grids in the parameter plane [34], [35].

1.4 TDS with small delays - survey for previous works

TDSs are rigorously studied in the fields of mathematics. Ryabov [36] introduced a family of special solutions for a class of linear differential equations with small delays and showed that every solution is asymptotic to some special solution as $t \to \infty$. Ryabov's result was improved by Driver [37], Jarník and Kurzweil [38]. A more precise asymptotic description was given by Arino and Pituk in [39]. For other related results on asymptotic integration and stability of linear differential equations with small delays, see the result of Arino, Győri and Pituk [40], and Győri and Pituk [41]. Faria and Huang [42] gave some improvements, and a generalisation to functional differential equations in Banach spaces. Inertial and slow manifolds for differential equations with small delays were studied by Chicone [43]. Results on minimal sets of a skew-product semiflow generated by scalar differential equations with small delay can be found in the work of Alonso, Obaya and Sanz [44]. Smith and Thieme [45] showed that nonlinear autonomous differential equations with small delay generate a monotone semiflow with respect to the exponential ordering, and the monotonicity has important dynamical consequences. For the effects of small delays on stability and control, see the paper by Hale and Verduyn Lunel [46].

The results in the above-listed papers show that if the delay is small, there are similarities between the delay differential equation and an ordinary differential equation. The description of the associated ordinary differential equation, in general, requires the knowledge of certain special solutions. Since, in most cases, the special solutions are not known, the above results are mainly of theoretical interest.

Illustrative example for the smallness condition domain presented in [47, Equation 1.9]: In control applications, the closed loop gain of the systems are near unity. If we consider the maximum gains in the equation referred above to be unity, the small delay domain yields as $0 < \tau < 0.278s$. Figure 1.1 shows the delay values that are considered "small". The maximum delay value $0.278s$ corresponds to realistic communication delay values in networked control systems, see e.g. [48], where the communication delay was always below $120ms$ using the low latency Dedicated Short Range communication Connectivity (DSRC).

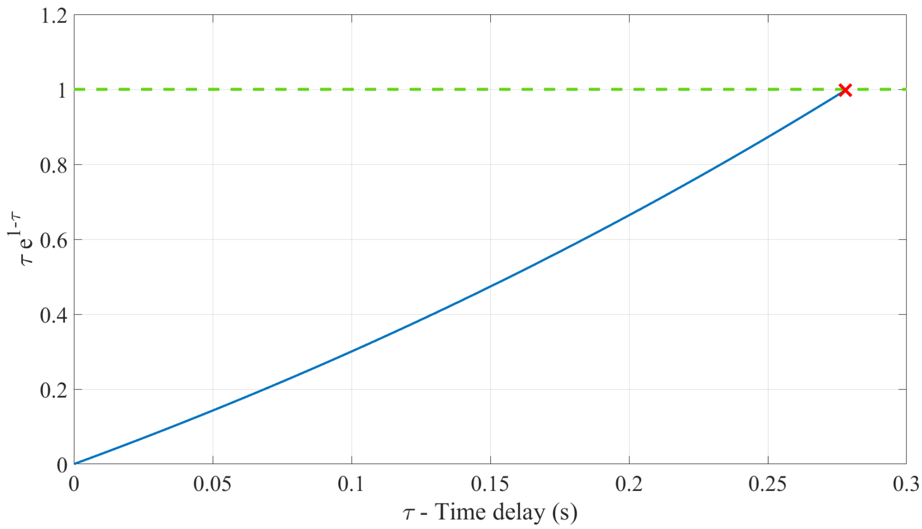

Figure 1.1 In a unit-gain system, the time delay is considered small if $\tau e^{1+\tau} < 1$ holds. The figure shows the $\tau e^{1+\tau}$ curve for delays $0s < \tau \leq 0.278s$.

1.5 Thesis summary of the contributions

The results are structured in three main parts. Chapter 2 is dedicated to the approximation of constant, point-wise delays in continuous time. A homogeneous TDS is given and it is shown that under a certain smallness condition, it can be approximated with an ODE, which has the same number of states. The approximation error converges to zero exponentially. An analytic equation is given to calculate the system matrix of the ODE. Furthermore, it is shown that an iterative method can also be used to give this state matrix. The convergence of this iteration is proven to be exponential. Next, the TDS system is extended with a bounded nonhomogeneous term, and an analytic solution and a numerical iteration is given to find the equivalent additive term for the approximating ODE which preserves the exponential approximating characteristics of our system. Finally, explicit conditions are provided for the detectability of the TDS based on the ODE, and an observer design procedure is devised based on the previously shown approximation method.

Chapter 3 deals with the approximation of discrete-time linear systems with time-delays. First, the homogeneous discrete-time Volterra type difference systems is studied with infinite delays and it is shown that the system is asymptotically equivalent to an ordinary difference equation under a smallness condition. An explicit relation is given to calculate the system matrix, and it is also shown that an iterative method can be used to approximate the system matrix of the original system containing delay. Both the iterative matrix equations and the approximation methods are shown to be exponentially convergent. The Volterra type difference system is set to a finite number of delays. The Volterra system is extended with a bounded nonhomogeneous term and an equivalent approximating ordinary difference system is given. Explicit and iterative relations are given to calculate the nonhomogeneous term of the approximating ordinary difference system based on the original extended Volterra system. Next, the shown methods are applied to the analysis of Multi-Agent Systems with delay present in the communication graph.

Chapter 4 deals with the approximation of continuous-time linear systems with distributed time delay. First, the existence of an approximating ODE is shown under a smallness condition. The trajectories of the original system converge to the trajectories of the ODE exponentially. Next, explicit and iterative methods are given for finding the system matrix of the ODE. The convergence of the iterative method is proven exponential. The original homogeneous delay system is extended with a bounded nonhomogeneous term and an explicit and iterative method is given to find the equivalent nonhomogeneous term of the approximating ODE which guarantees that the trajectories still converge. Finally, the provided approximation method is used to check the stabilizability of the delayed system based on the generated ODE. The approximation method is also used for the design of a controller for the delayed system based on the approximating ODE.

Under appropriate smallness conditions, the provided approximation method has the following attributes:

- ensures that the original delayed system and the delay free approximating system are asymptotically equivalent.
- gives a numerical method to compute the dominant eigenvalues of the original delayed system.
- provides numerical algorithms for the state and input matrices of the delay free approximating system, which are important for system and control theoretical applications.
- although the approximation method is not suitable for the numerical approximation of the solutions, it can be used to describe the asymptotic behaviour of the solutions of the original delayed system.

Chapter 2

Approximation of continuous-time linear time delay systems with point-wise time delays

2.1 Abstract

In this chapter, it is shown that if a certain smallness condition between the delay and the system gains holds, then a continuous-time Linear Time-Invariant (LTI) system with pointwise delay can be approximated by a linear delay-free system. Furthermore, it is shown that the proposed method can be explored to analyse the detectability of the delay system. A simplified observer design method is proposed for the addressed class of delay systems. This chapter is based on [49*, 15*, 50*, 51*, 52*].

2.2 Literature survey

The description of the associated ordinary differential equation, in general, requires the knowledge of certain special solutions. Since, in most cases, the special solutions are not known, the above results are mainly of theoretical interest.

For linear systems, without time delay, there are well-known observer design methods such as the Luenberger observer, the Kalman filter, the H_∞ observer, the sliding-mode observer, etc. During the last few decades, TDS observers have been widely contemplated. The observability of delay systems was analysed, e.g. by Sun Yi et al. [53], and Emilia Fridman [54]. Basin et al. [55] presented an optimal filtering method for linear systems containing state delays.

Pakzad [56] developed a Kalman filter method for linear TDS with state, and output delays. An observer was presented by Safarinejadian et al. [57] for discrete-time linear systems with unit time delay. A state estimator was devised by Tan et. al. [58] for TDS with Markov-jump parameters. Zheng et al. [59] proposed a PD type H_∞ observer for linear systems with state delay . Huong [60] designed an observer for systems with state and output delays based on state coordinate transform, and Luenberger observer. Chou and Cheng [61] presented an optimal observer for linear TDS with state delay based on evolutionary optimisation. Targui et al. [62] developed a state observer for linear and Lipschitz nonlinear systems with bounded and variable delayed outputs. Mohajerpoor [63] proposed a delay dependent functional partial state observer for linear TDS with state and input delays of uncertain value.

2.3 Approximation of the homogeneous part

Consider the system (1.12), with $A_\tau \neq O_n$, and $\tau > 0$. The eigenvalues $\lambda \in \mathbb{C}$ of (1.12) satisfy the characteristic equation (1.13), which in general has infinitely many solutions. The n rightmost eigenvalues of the characteristic equation will be called *dominant eigenvalues* . Throughout the chapter it is assumed that the relation

$$\|A_\tau\|\tau e^{1+\|A_0\|\tau} < 1 \tag{2.1}$$

holds, which may be viewed as a *smallness condition* on the delay τ.

It will be shown that if (2.1) holds, then the system (1.12) is asymptotically equivalent to the ODE

$$\dot{\underline{x}}(t) = M\underline{x}(t), \tag{2.2}$$

with $M \in \mathbb{R}^{n\times n}$ being the unique solution of the matrix equation

$$M = A_0 + A_\tau e^{-\tau M}, \tag{2.3}$$

such that

$$\|M\| < \nu_0, \quad \text{where } \nu_0 = -\frac{1}{\tau}\ln(\|A_\tau\|\tau) > 0. \tag{2.4}$$

Furthermore, the system matrix M in (2.3) can be written as a limit of successive approximations

$$M = \lim_{k\to\infty} M_k, \tag{2.5}$$

where

$$M_0 = O_n \quad \text{and} \quad M_{k+1} = A_0 + A_\tau e^{-\tau M_k} \qquad \text{for } k = 0, 1, \ldots. \tag{2.6}$$

The convergence in (2.5) is exponential and an estimate is given for the approximation error $\|M - M_k\|$. It will be shown that those characteristic roots of (1.12) which lie in the half-plane $\Re(\lambda) > -\nu_0$, coincide with the eigenvalues of matrix M. As a consequence, the above dominant characteristic roots of (1.12) can be approximated by the eigenvalues of M_k. An explicit estimate is given for the approximation error, which shows that the convergence of the eigenvalues of M_k to the dominant characteristic roots of (1.12) is exponentially fast.

2.3.1 Solution of the matrix equation and its approximation

First, some lemmas are needed for the proof of the existence and uniqueness of the solution of the matrix equation (2.3) satisfying (2.4).

Lemma 2.3.1. *Let $P, Q \in \mathbb{R}^{n\times n}$, and $\gamma = \max\{\|P\|, \|Q\|\}$. Then*

$$\|P^k - Q^k\| \leq k\gamma^{k-1}\|P - Q\| \quad \textit{for } k = 1, 2, \ldots. \tag{2.7}$$

Proof. It will be shown by induction on k that

$$P^k - Q^k = \sum_{j=0}^{k-1} P^j(P - Q)Q^{k-1-j} \qquad \text{for } k = 1, 2, \ldots. \tag{2.8}$$

Evidently, the relation (2.8) holds for $k = 1$. Suppose that (2.7) also holds for some $k \in \mathbb{N}$. Then

$$P^{k+1} - Q^{k+1} = P^k(P-Q) + (P^k - Q^k)Q =$$
$$= P^k(P-Q) + \left(\sum_{j=0}^{k-1} P^j(P-Q)Q^{k-1-j}\right)Q = \sum_{j=0}^{k} P^j(P-Q)Q^{k-j}.$$

Thus, by induction on k, (2.7) holds.

From (2.7) we have

$$\|Q^k - P^k\| \le \sum_{j=0}^{k-1} \|P\|^j \|Q-P\| \|Q\|^{k-1-j} \le \|P-Q\| \sum_{j=0}^{k-1} \gamma^j \gamma^{k-1-j} = k\gamma^{k-1}\|P-Q\|$$

for $k = 1, 2, \ldots$. □

Using Lemma 2.3.1, the following result can be proven about the distance of two matrix exponentials.

Lemma 2.3.2. *Let $P, Q \in \mathbb{R}^{n\times n}$ and $\gamma = \max\{\|P\|, \|Q\|\}$. Then*

$$\|e^P - e^Q\| \le e^\gamma \|P-Q\|. \tag{2.9}$$

Proof. By the definition of matrix exponential, we have

$$e^P - e^Q = \sum_{k=0}^{\infty} \frac{1}{k!}(P^k - Q^k).$$

From this, by application of Lemma 2.3.1, we find

$$\|e^P - e^Q\| \le \sum_{k=0}^{\infty} \frac{\|P^k - Q^k\|}{k!} \le \|P-Q\| \sum_{k=1}^{\infty} \frac{k\gamma^{k-1}}{k!} =$$
$$= \|P-Q\| \sum_{k=1}^{\infty} \frac{\gamma^{k-1}}{(k-1)!} = e^\gamma \|P-Q\|$$

which proves (2.9). □

Some properties of the scalar equation

$$\nu = \|A_0\| + \|A_\tau\| e^{\tau\nu} \tag{2.10}$$

are also needed.

Lemma 2.3.3. *Suppose that the smallness condition* (2.1) *holds. If we let $\nu_0 = -\frac{1}{\tau}\ln(\|A_\tau\|\tau)$, then $\nu_0 > 0$, and* (2.10) *has a unique root $\nu_1 \in (0, \nu_0)$. Moreover,*

$$\|A_0\| + \|A_\tau\| e^{\tau\nu} < \nu \quad \textit{for } \nu \in (\nu_1, \nu_0], \tag{2.11}$$

and

$$\tau \|A_\tau\| e^{\tau\nu} < 1 \quad \textit{for } \nu < \nu_0. \tag{2.12}$$

Proof. From (2.1), we have $\|A_\tau\|\tau < e^{-1-\|A_0\|\tau} < 1$ which implies that $\ln(\|A_\tau\|\tau) < 0$. Hence $\nu_0 > 0$. Let

$$f(\nu) = \nu - \|A_0\| - \|A_\tau\| e^{\nu\tau},$$

for $\nu \in \mathbb{R}$. We have

$$f'(\nu) = 1 - \|A_\tau\|\tau e^{\nu\tau} \qquad \text{and} \qquad f''(\nu) = -\|A_\tau\|\tau^2 e^{\nu\tau}$$

for $\nu \in \mathbb{R}$. It is easily seen that $f'(\nu) = 0$ if and only if $\nu = \ln(\|A_\tau\|\tau)/\tau = \nu_0$. Furthermore, (2.1) is equivalent to $f(\nu_0) < 0$. Since $f''(\nu) < 0$ on $\nu \in \mathbb{R}$, $f'(\nu)$ is strictly descreasing on $\mathbb{R}$. In particular, $f'(\nu) > f'(\nu_0) = 0$ for $\nu < \nu_0$. Therefore (2.12) holds and f is strictly increasing on $(-\infty, \nu_0]$. This, together with $f(0) < 0$ and $f(\nu_0) > 0$, implies that f and hence (2.10) has a unique root $\nu_1 \in (0, \nu_0)$. Since f is strictly increasing on $[\nu_1, \nu_0]$, we have that $f(\nu) > f(\nu_1) = 0$ for $\nu \in (\nu_1, \nu_0]$. Thus, (2.11) holds.

□

Now the theorem for the existence and uniqueness of the solution of the matrix equation can be formulated and proved.

Theorem 2.3.4. *Suppose* (2.1) *holds. Then* (2.3) *has a unique solution* $M \in \mathbb{R}^{n\times n}$ *such that* (2.4) *holds.*

Proof. If (2.1) holds, then, by Lemma 2.3.3, the equation (2.10) has a unique solution $\nu_1 \in (0, \nu_0)$, with ν_0 given by (2.4).

Let $\nu \in [\nu_1, \nu_0)$ be fixed. Define

$$F(M) = A_0 + A_\tau e^{-M\tau} \quad \text{for } M \in \mathbb{R}^{n\times n}, \tag{2.13}$$

and

$$\mathcal{S} = \{M \in \mathbb{R}^{n\times n} \mid \|M\| \le \nu\}. \tag{2.14}$$

Clearly, $\mathcal{S}$ is a nonempty and closed subset of $\mathbb{R}^{n\times n}$. By virtue of Lemma 2.3.3, we have for $M \in \mathcal{S}$,

$$\|F(M)\| \le \|A_0\| + \|A_\tau\| e^{\|M\|\tau} \le \|A_0\| + \|A_\tau\| e^{\nu\tau} \le \nu. \tag{2.15}$$

Thus F maps $\mathcal{S}$ into itself. Let $M_1, M_2 \in \mathcal{S}$. By the application of Lemma 2.3.2, we obtain

$$\begin{aligned}\|F(M_1) - F(M_2)\| = \|A_\tau(e^{-M_1\tau} - e^{-M_2\tau})\| \le \|A_\tau\|\|e^{-M_1\tau} - e^{-M_2\tau}\| \\ \le \underbrace{\|A_\tau\|\tau e^{\nu\tau}}_{\kappa<1}\|M_1 - M_2\|.\end{aligned}$$

In view of this, $F : \mathcal{S} \to \mathcal{S}$ is a contraction and by the Banach's fixed point theorem (Appendix B.1.2), there exists a unique $M \in \mathcal{S}$ such that $M = F(M)$. Since $\nu \in [\nu_1, \nu_0)$ was arbitrary, this completes the proof. □

In the next theorem, it is shown that the unique solution of the matrix equation (2.3) with property (2.4) can be written as a limit of successive approximations M_k defined by (2.6) and we give an estimate for the approximation error.

Theorem 2.3.5. *Suppose* (2.1) *holds and let* $M \in \mathbb{R}^{n\times n}$ *be the solution of* (2.3) *with property* (2.4). *If* $\{M_k\}_{k=0}^{\infty}$ *is the sequence of matrices defined by* (2.6), *then*

$$\|M_k\| \le \nu_1 \quad \text{for } k = 0, 1, 2, \cdots, \tag{2.16}$$

and

$$\|M - M_k\| \le \nu_1 \kappa^k \quad \text{for } k = 0, 1, 2, \cdots, \tag{2.17}$$

where ν_1 *is the unique root of* (2.10) *in the interval* $(0, \nu_0)$, *and* $\kappa = \|A_\tau\|\tau e^{\nu_1 \tau} < 1$ *(see* (2.12)*).*

Proof. Note that $M_{k+1} = F(M_k)$ for $k = 0, 1, 2, \ldots$, where F is defined by (2.13). Taking $\nu = \nu_1$ in the proof of Theorem 2.3.4, we find that $\|M\| \le \nu_1$. Also, from (2.14) and (2.15), we obtain that $\|M_k\| \le \nu_1$ for $k = 0, 1, 2, \ldots$. From this, and equations (2.3) and (2.6), by the application of Lemma 2.3.2, we obtain for $K \ge 0$,

$$\begin{aligned}\|M - M_{k+1}\| = \|A_\tau(e^{-M\tau} - e^{-M_k\tau})\| \le \|A_\tau\|\|e^{-M\tau} - e^{-M_k\tau}\| \\ \le \underbrace{\|A_\tau\|\tau e^{\nu_1\tau}}_{\kappa} \|M - M_k\|.\end{aligned}$$

From the last inequality, it follows by induction on k that

$$\|M - M_k\| \le \kappa^k \|M - M_0\| = \kappa^k \|M\| \le \kappa^k \nu_1$$

for $k = 0, 1, 2, \ldots$. □

2.3.2 Dominant eigenvalues and eigensolutions

Let us summarize some facts from the theory of linear autonomous delay differential equations (see [1], [64]). By an eigenvalue of (1.12), we mean an eigenvalue of the generator of the solution semigroup (see [1], [64] for details). It is known that $\lambda \in \mathbb{C}$ is an eigenvalue of (1.12) if and only if λ is a root of the characteristic equation (1.13). Moreover, for every $\beta \in \mathbb{R}$, Eq. (1.12) has only finite number of eigenvalues with $\Re(\lambda) > \beta$. By an *entire solution* of (1.12), we mean a differentiable function $\underline{x} : \mathbb{R} \to \mathbb{R}^n$ satisfying (1.12) for all $t \in \mathbb{R}$. To each eigenvalue λ of (1.12), there correspond nontrivial entire solutions of the form $\underline{p}(t)e^{\lambda t}$, $t \in \mathbb{R}$, where $\underline{p}(t)$ is a $\mathbb{R}^n$-valued polynomial in t. Such solutions are sometimes called eigensolutions corresponding to λ.

As a preparation, three lemmas are established . First it is shown that if M is a solution of the matrix equation (2.3), then every solution of the ODE (2.2) is an entire solution of the DDE (1.12).

Lemma 2.3.6. *Let* $M \in \mathbb{R}^{n\times n}$ *be a solution of* (2.3). *Then for every* $\underline{v} \in \mathbb{R}^n$, $\underline{x}(t) = e^{Mt}\underline{v}$, $t \in \mathbb{R}$, *is an entire solution of* (1.12).

Proof. Since $e^P e^Q = e^{P+Q}$ whenever $P, Q \in \mathbb{R}^{n\times n}$ commute, from (2.3), we find that

$$\begin{aligned}\dot{\underline{x}}(t) = Me^{Mt}\underline{v} = (A_0 + A_\tau e^{-\tau M})e^{Mt}\underline{v} = \\ = A_0 e^{Mt}\underline{v} + A_\tau e^{M(t-\tau)}\underline{v} = A_0\underline{x}(t) + A_\tau \underline{x}(t-\tau), \qquad \text{for all } t \in \mathbb{R}.\end{aligned}$$

□

In the following lemma, the uniqueness of entire solutions of the DDE (1.12) with an appropriate exponential growth as $t \to -\infty$ is proven.

Lemma 2.3.7. *Suppose* (2.1) *holds. If* $\underline{x}_1(t), \underline{x}_2(t)$ *are entire solutions of* (1.12) *with* $\underline{x}_1(0) = \underline{x}_2(0)$ *and such that*

$$\sup_{t\le 0} \|\underline{x}_j(t)\| e^{\nu_0 t} < \infty, \quad j = 1, 2, \tag{2.18}$$

with ν_0 *as in* (2.4)*, then* $\underline{x}_1(t) = \underline{x}_2(t)$ *identically on* $\mathbb{R}$*.*

Proof. Define

$$c = \sup_{t\le 0} \|\underline{x}_1(t) - \underline{x}_2(t)\| e^{\nu_0 t}.$$

By virtue of (2.18), we have that $0 \le c < \infty$. From (1.12), we find for $t \le 0$,

$$\underline{x}_j(t) = \underline{x}_j(0) - A_0 \int_t^0 \underline{x}_j(s) \mathrm{d}s - A_\tau \int_t^0 \underline{x}_j(s-\tau) \mathrm{d}s, \quad j = 1, 2.$$

From this, taking into account that $\underline{x}_1(0) = \underline{x}_2(0)$, we obtain for $t \le 0$,

$$\begin{aligned} \|\underline{x}_1(t) - \underline{x}_2(t)\| &\le \|A_0\| \int_t^0 \|\underline{x}_1(s) - \underline{x}_2(s)\| \mathrm{d}s + \|A_\tau\| \int_t^0 \|\underline{x}_1(s-\tau) - \underline{x}_2(s-\tau)\| \mathrm{d}s \le \\ &\le \|A_0\| c \int_t^0 e^{-\nu_0 s} \mathrm{d}s + \|A_\tau\| c \int_t^0 e^{-\nu_0 (s-\tau)} \mathrm{d}s = \\ &= c(\|A_0\| + \|A_\tau\| e^{\nu_0 \tau}) \int_t^0 e^{-\nu_0 s} \mathrm{d}s \le c \frac{\|A_0\| + \|A_\tau\| e^{\nu_0 \tau}}{\nu_0} e^{-\nu_0 t}. \end{aligned}$$

The last inequality implies for $t \le 0$,

$$\|\underline{x}_1(t) - \underline{x}_2(t)\| e^{\nu_0 t} \le c \underbrace{\frac{\|A_0\| + \|A_\tau\| e^{\nu_0 \tau}}{\nu_0}}_{\varkappa}.$$

Hence $c \le \varkappa c$. By virtue of (2.11), we have that $\varkappa < 1$. Hence $c = 0$, and $\underline{x}_1(t) = \underline{x}_2(t)$ for $t \le 0$. The uniqueness theorem [1, Chap.2, Theorem 2.3] implies that $\underline{x}_1(t) = \underline{x}_2(t)$, $\forall t \in \mathbb{R}$. □

Now it is shown that those entire solutions of (1.12) which satisfy the growth condition

$$\sup_{t\le 0} \|\underline{x}(t)\| e^{\nu_0 t} < \infty \tag{2.19}$$

with ν_0 as in (2.4) coincide with the solutions of the ODE (2.2).

Lemma 2.3.8. *Suppose* (2.1) *holds. Then, for every* $\underline{v} \in \mathbb{R}^n$, (1.12) *has exactly one entire solution* $\underline{x}(t)$ *with* $\underline{x}(0) = \underline{v}$ *and satisfying* (2.19) *given by*

$$\underline{x}(t) = e^{Mt} \underline{v} \quad \text{for } t \in \mathbb{R}, \tag{2.20}$$

where $M \in \mathbb{R}^{n\times n}$ *is the solution of* (2.3) *with property* (2.4)*.*

Proof. By Lemma 2.3.6, $\underline{x}(t)$ defined by (2.20) is an entire solution of (1.12). Moreover, from (2.4) and (2.20), we find for $t \le 0$,

$$\|\underline{x}(t)\| \le e^{\|M\||t|} \|\underline{v}\| \le e^{\nu_0 |t|} \|\underline{v}\| = e^{-\nu_0 t} \|\underline{v}\|$$

Hence

$$\sup_{t\le 0} \|\underline{x}(t)\| e^{\nu_0 t} \le \|v\| < \infty.$$

Thus, $\underline{x}(t)$ given by (2.20) is an entire solution of (1.12) with $\underline{x}(0) = \underline{v}$ and satisfying (2.19). The uniqueness follows from Lemma 2.3.7. □

The following theorem shows that under the smallness condition (2.1) the eigenvalues of (1.12) with $\Re(\lambda) > -\nu_0$ coincide with eigenvalues of matrix M from Theorem 2.3.4 and the corresponding eigensolutions satisfy the ordinary differential equation (2.2).

Theorem 2.3.9. *Suppose* (2.1) *holds so that* $\nu_0 = -\ln(\|A_\tau\|\tau)/\tau > 0$, *and define*

$$\Lambda = \{\lambda \in \mathbb{C} \mid \det D(\lambda) = 0, \Re(\lambda) > -\nu_0\}.$$

Let $M \in \mathbb{R}^{n\times n}$ *be the unique solution of* (2.3) *satisfying* (2.4). *Then* $\Lambda = \sigma(M)$. *Moreover, for every* $\lambda \in \Lambda$, *Equations* (1.12) *and* (2.2) *have the same eigensolutions corresponding to* λ.

Proof. Suppose that $\lambda \in \Lambda$. Since $\det D(\lambda) = 0$, there exists a nonzero vector $\underline{v} \in \mathbb{R}^n$ such that $D(\lambda)\underline{v} = 0$ and hence $\underline{x}(t) = \mathrm{e}^{\lambda t}\underline{v}$, $t \in \mathbb{R}$, is an entire solution of (1.12). $\Re(\lambda) > -\nu_0$, so for $t \le 0$ we have

$$\|\underline{x}(t)\| = |\mathrm{e}^{\lambda t}|\|\underline{v}\| = \mathrm{e}^{\Re(\lambda)t}\|\underline{v}\| \le \mathrm{e}^{-\nu_0 t}\|\underline{v}\|,$$

which implies (2.19). Thus, $\underline{x}(t) = \mathrm{e}^{\lambda t}\underline{v}$ is an entire solution of (1.12) with $\underline{x}(0) = \underline{v}$ and satisfying (2.19). By Lemma 2.3.8 , we have that $\mathrm{e}^{\lambda t}\underline{v} = \mathrm{e}^{Mt}\underline{v}$ for $t \in \mathbb{R}$. Hence

$$\frac{\mathrm{e}^{\lambda t} - 1}{t}\underline{v} = \frac{\mathrm{e}^{Mt} - I_n}{t}\underline{v} \quad \text{for } t \in \mathbb{R}^*.$$

Letting $t \to 0$, we obtain $\lambda\underline{v} = M\underline{v}$. This proves that $\Lambda \subset \sigma(M)$.

Now suppose $\lambda \in \sigma(M)$. Then there exists a nonzero vector $\underline{v} \in \mathbb{R}^n$ such that $M\underline{v} = \lambda\underline{v}$. According to Lemma 2.3.6, $\underline{x}(t) = \mathrm{e}^{\lambda t}\underline{v} = \mathrm{e}^{Mt}\underline{v}$ is an entire solution of (1.12). Hence $D(\lambda)\underline{v} = \underline{0}$ which implies that $\det D(\lambda) = 0$. In order to prove that $\lambda \in \Lambda$, it remains to show that $\Re(\lambda) > -\nu_0$. It is well-known that $\rho(M) \le \|M\|$. This, together with (2.4), yields

$$|\Re(\lambda)| \le |\lambda| \le \rho(M) \le \|M\| < \nu_0.$$

Therefore $\Re(\lambda) > -\nu_0$, which proves that $\sigma(M) \subset \Lambda$.

Let $\lambda \in \Lambda = \sigma(M)$. By Lemma 2.3.6, every eigensolution of the ordinary differential equation (2.2) corresponding to λ is an eigensolution of the delay differential equation (1.12). Now suppose that $\underline{x}(t)$ is an eigensolution of the delay differential equation (1.12) corresponding to λ. Then $\underline{x}(t) = \underline{p}(t)\mathrm{e}^{\lambda t}$, where $\underline{p}(t)$ is a $\mathbb{R}^n$ -valued polynomial in t. If m is the degree of the polynomial p, then there exists $k > 0$ such that

$$\|\underline{p}(t)\| \le k(1 + |t|^m) \quad \text{for } t \in \mathbb{R}.$$

Since $\Re(\lambda) > -\nu_0$, we have that $\epsilon = \Re(\lambda) + \nu_0 > 0$. From this, we find for $t \le 0$,

$$\|\underline{x}(t)\| = \|\underline{p}(t)\||\mathrm{e}^{\lambda t}| = \|\underline{p}(t)\|\mathrm{e}^{\Re(\lambda)t} \le k(1 + |t|^m)\mathrm{e}^{\Re(\lambda)t} = k(1 + |t|^m)\mathrm{e}^{\epsilon t}\mathrm{e}^{-\nu_0 t}.$$

Hence

$$\|\underline{x}(t)\|\mathrm{e}^{\nu_0 t} \le k(1 + |t|^m)\mathrm{e}^{\epsilon t} \to 0 \quad \text{as } t \to -\infty.$$

Thus, $\underline{x}(t)$ is an entire solution of (1.12) satisfying the growth condition (2.19). By Lemma 2.3.8, $\underline{x}(t)$ is a solution of the ODE (2.2). □

Remark 2.3.10. Theorem 2.3.9 has an interesting consequence. If a delayed system in the form of (1.12) with n state variables satisfies the smallness condition (2.1), then the n rightmost eigenvalues of the delayed system counting multiplicities are situated in the half plane $\Re(\lambda) > -\nu_0$ and they coincide with the eigenvalues of the delay free approximating system (2.2).

2.3.3 Asymptotic equivalence

The following result from the monograph by Diekmann et al. [64] gives an asymptotic description of the solutions of (1.12) in terms of the eigensolutions.

Proposition 2.3.11. *[64, Chap. I, Theorem 5.4] Let $\underline{x} : [-\tau, \infty) \to \mathbb{R}^n$ be a solution of* (1.12) *corresponding to some continuous initial function $\theta : [-\tau, 0] \to \mathbb{R}^n$. For any $\gamma \in \mathbb{R}$ such that* $\det(D(\lambda)) = 0$ *has no roots on the vertical line $\Re(\lambda) = \gamma$, we have the asymptotic expansion*

$$\underline{x}(t) = \sum_{j=1}^{l} \underline{p}_j(t)\mathrm{e}^{\lambda_j t} + \underline{o}(\mathrm{e}^{\gamma t}) \quad \text{as } t \to \infty, \tag{2.21}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_l$ are the finitely many roots of the characteristic equation (1.13) *with real part greater than γ and $\underline{p}_j(t)$ are $\mathbb{R}^n$-valued polynomials in t of order less than the multiplicity of λ_j as a zero of* $\det(D(\lambda))$.

Now the main result can be formulated about the asymptotic equivalence of (1.12) and (2.2).

Theorem 2.3.12. *Suppose* (2.1) *holds so that $\nu_0 = -\ln(\|A_\tau\|\tau)/\tau > 0$. Let $M \in \mathbb{R}^{n\times n}$ be the solution of the matrix equation* (2.3) *satisfying* (2.4). *Then the following statements are valid:*

1. *Every solution of the ODE* (2.2) *is an entire solution of the DDE* (1.12).
2. *For every solution $\underline{x} : [-\tau, \infty) \to \mathbb{R}^n$ the DDE* (1.12) *corresponding to some continuous initial function $\underline{\theta} : [-\tau, 0] \to \mathbb{R}^n$, there exists a solution $\underline{\hat{x}}(t)$ of the ODE* (2.2) *such that*

$$\underline{x}(t) = \underline{\hat{x}}(t) + \underline{o}(\mathrm{e}^{-\nu_0 t}) \quad \text{as } t \to \infty. \tag{2.22}$$

Proof. Conclusion 1 follows from Lemma 2.3.1. Conclusion 2 shall be proved by applying Proposition 2.3.11 with $\gamma = -\nu_0$. It must be veried that the characteristic equation (1.13) has no root on the vertical line $\Re(\lambda) = \nu_0$. Suppose for contradition that there exists $\lambda \in \mathbb{C}$ such that $\det(D(\lambda)) = 0$ and $\Re(\lambda) = \nu_0$. Then there exists a nonzero vector $\underline{v} \in \mathbb{R}^n$ such that $D(\lambda)\underline{v} = \underline{0}$ and hence $\lambda\underline{v} = A_0\underline{v} + A_\tau\mathrm{e}^{-\lambda\tau}\underline{v}$. From this, we find that

$$|\lambda|\|\underline{v}\| \le \|A_0\|\|\underline{v}\| + \|A_\tau\|\|\mathrm{e}^{-\lambda\tau}\underline{v}\| = \|A_0\|\|\underline{v}\| + \|A_\tau\||\mathrm{e}^{-\lambda\tau}|\|\underline{v}\| = \\ = (\|A_0\| + \|A_\tau\|\mathrm{e}^{-\tau\Re(\lambda)})\|\underline{v}\| = (\|A_0\| + \|A_\tau\|\mathrm{e}^{\nu_0\tau})\|\underline{v}\| \tag{2.23}$$

Hence $|\lambda| \le \|A_0\| + \|A_\tau\|\mathrm{e}^{\nu_0\tau}$, which together with (2.11), yields

$$\nu_0 = |\Re(\lambda)| \le |\lambda| \le \|A_0\| + \|A_\tau\|\mathrm{e}^{\nu_0\tau} < \nu_0,$$

a contradiction. Thus, Proposition 2.3.11 can be applied with $\gamma = -\nu_0$, which implies that the asymptotic relation (2.22) holds with

$$\underline{\hat{x}}(t) = \sum_{j=1}^{l} \underline{p}_j(t) e^{\lambda_j t}, \tag{2.24}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_l$ are thoise eigenvalues of (1.12) which have real part greater than $-\nu_0$ and $\underline{p}_j(t)$ are $\mathbb{R}^n$-valued polynomials in t. According to Theorem 2.3.9, the eigensolutions of (1.12) corresponding to the eigenvalues with real part greater than $-\nu_0$ are solution of the ODE (2.2). Hence $\underline{\hat{x}}(t)$ given by (2.24) is a solution of (2.2). □

2.3.4 Approximation of dominant eigenvalues

The following result will be needed about the distance of the eigenvalues of two matrices in terms of the norm of their difference due to Bhatia, Elsner and Krause [65].

Proposition 2.3.13. *[65, Theorem 3] Let $P, Q \in \mathbb{R}^{n\times n}$ and $\gamma = \max\{\|P\|, \|Q\|\}$. Then the eigenvalues of P and Q can be enumerated as $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_n$ in such a way that*

$$\max_{1\le j\le n} |\lambda_j - \mu_j| \le 4 \cdot 2^{-\frac{1}{n}} n^{\frac{1}{n}} (2\gamma)^{1-\frac{1}{n}} \|P - Q\|^{\frac{1}{n}}. \tag{2.25}$$

Recall that the dominant eigenvalues of (1.12) are those roots of the characteristic equation (1.13) which have real part greater than $-\nu_0$. According to Theorem 2.3.9, if (2.1) holds, then the dominant eigenvalues of (1.12) coincide with the eigenvalues of M, the unique solution of the matrix equation (2.3) satisfying (2.4). By Theorem 2.3.5, M can be approximated by the sequence of matrices $\{M_k\}_{k=0}^{\infty}$ defined by (2.6). As a consequence, the dominant eigenvalues of the delay differential equation (1.12) can be approximated by the eigenvalues of M_k. The explicit estimate (2.17) for the distance $\|M - M_k\|$, combined with Proposition 2.3.13, yields the following result.

Theorem 2.3.14. *Suppose* (2.1) *holds so that the dominant eigenvalues of* (1.12) *coincide with the eigenvalues $\lambda_1, \ldots, \lambda_n$ of matrix M from Theorem 2.3.4 (see Theorem 2.3.9). If $\{M_k\}_{k=0}^{\infty}$ is the sequence of matrices defined by* (2.6)*, then the eigenvalues $\lambda_1^{[k]}, \ldots, \lambda_n^{[k]}$ of M_k can be renumbered such that*

$$\max_{1\le j\le n} |\lambda_j - \lambda_j^{[k]}| \le 8 \cdot 4^{-\frac{1}{n}} n^{\frac{1}{n}} \nu_1 \kappa^{\frac{k}{n}}, \tag{2.26}$$

where ν_1 and κ have the meaning from Theorem 2.3.5.

Since $\kappa < 1$, the explicit error estimate (2.26) in Theorem 2.3.14 shows that under the smallness condition (2.1) the eigenvalues of M_k converge to the dominant eigenvalues of the DDE (1.12) at an exponential rate as $k \to \infty$.

Theorem 2.3.14 can be illustrated by the following simple two-dimensional example.

Example 2.3.1. Consider the DDE in the form of (1.12), with $n = 2$, $\tau = 1$ and system matrices

$$A_0 = \begin{pmatrix} -0.14 & 0 \\ 0 & -0.14 \end{pmatrix}, A_\tau = \begin{pmatrix} 0 & 0.14 \\ 0.14 & 0 \end{pmatrix}.$$

If $\|\cdot\| = \|\cdot\|_1$, then $\|A_0\| = \|A_\tau\| = 0.14$ and hence the assumption (2.1) is satisfied with $\tau\|A_\tau\|e^{1+\tau\|A_0\|} = 0.14e^{1.14} \approx 0.43775 < 1$. The characteristic equation from (1.13) has the form $\det D(\lambda) = \lambda^2 + 0.28\lambda + 0.0196 - 0.0196e^{-2\lambda}$. It follows by easy numerical calculations that the values of the quantities ν_1 and κ from Theorem 2.3.14 are $\nu_1 \approx 0.33588$ and $\kappa \approx 0.19588$. By the application of Theorems 2.3.9 and 2.3.14, it can be concluded that in the region $\Re(\lambda) > -\nu_0$ the characteristic equation $\det D(\lambda) = 0$ has exactly two roots λ_1 and λ_2. Furthermore, the eigenvalues $\lambda_1^{[k]}$ and $\lambda_2^{[k]}$ of the successive approximations M_k given by (2.6) can be renumbered such that

$$\max_{j=1,2} |\lambda_j - \lambda_j^{[k]}| \leq \epsilon_k, \quad \text{where } \epsilon_k \approx 1.90002 \cdot 0.19588^{k/2}. \tag{2.27}$$

The roots of the characteristic equation $\det D(\lambda) = 0$ satisfying $\Re(\lambda) > -\nu_0$ are $\lambda_1 = 0$ and $\lambda_2 = -0.3358849196$. The approximations $\lambda_1^{[k]}$ and $\lambda_2^{[k]}$ of λ_1 and λ_2 were computed in MATLAB (see Table 4.1). The numerical results are in full agreement with the error estimate (3.40).

Table 2.1 Approximation of the characteristic roots in Example 2.3.1

k	$\lambda_1^{[k]}$	$\lambda_2^{[k]}$	$\lvert\lambda_1 - \lambda_1^{[k]}\rvert$	$\lvert\lambda_2 - \lambda_2^{[k]}\rvert$	ϵ_l
1	0	0	0	0.3359	0.8409
5	0	-0.3355	0	$0.4059e-3$	$0.3227e-1$
10	0	-0.3359	0	$0.1170e-6$	$0.5479e-3$
15	0	-0.3359	0	0	$0.9304e-5$
20	0	-0.3359	0	0	$0.1580e-6$
25	0	-0.3359	0	0	$0.2600e-8$

The following example shows the properties of the scalar function from the proof of Lemma 2.3.3 applied to Example 2.3.1.

Example 2.3.2. Consider a system in the form (1.12), with $A_0 = \begin{pmatrix} -0.6082 & 0.3159 \\ 0.0887 & -0.8846 \end{pmatrix}$, $A_\tau = \begin{pmatrix} -0.7429 & 0.3947 \\ 0.2815 & 0.5485 \end{pmatrix}$. The appropriate scalar function and its derivative are written as

$$f(\nu) = \nu - 1 - 0.8413e^{0.27\nu}$$
$$f'(\nu) = 1 - 0.2272e^{0.27\nu}.$$

Solving $f'(\nu_0) = 0$ in the region $\mathbb{R}_+^*$ gives $\nu_0 = 5.4886$. Solving $f(\nu_1) = 0$ in the interval $\nu_1 \in (0, \nu_0)$ gives $\nu_1 = 2.7841$.

Figure 2.1 shows the properties and the zero points of the scalar function and its derivative together with the maximum point of f.

Remark 2.3.15. If the original TDS has a constant initial function $\underline{x}(t) = \underline{x}_0$, $\forall t \in [-\tau, 0]$, the solutions of the approximating system of ODEs coincide with Pituk's special solutions [66] that corresponds to the initial value $\underline{x}_0$. If the initial function of the the original TDS is not constant, then the initial condition of the approximating system of ODEs can be given using the adjoint equation of the original TDS as shown in [47].

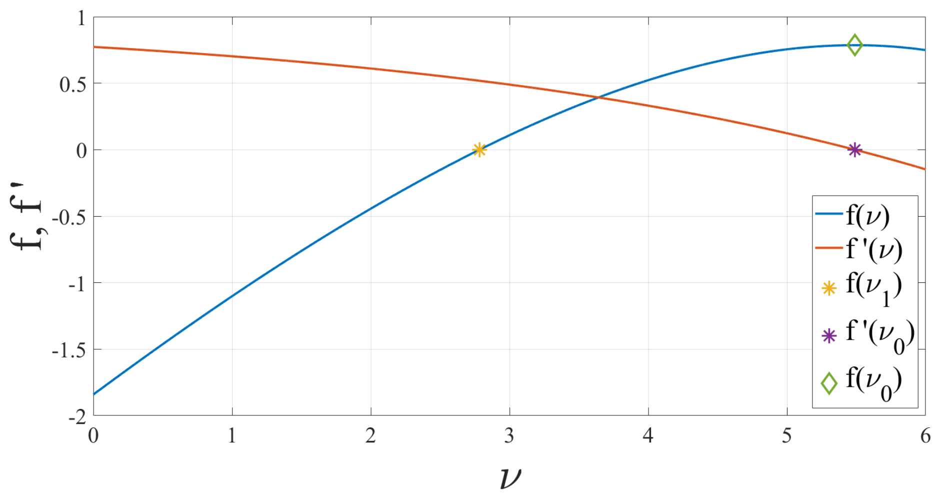

Figure 2.1 The function $f(\nu)$ and its derivative.

2.4 Extension to non-homogeneous systems

In this section, it is shown that the previously presented approximation method for homogeneous systems can be extended to TDS with non-homogeneous terms.

Theorem 2.4.1. *Consider the system*

$$\dot{\underline{x}}(t) = A_0\underline{x}(t) + A_\tau\underline{x}(t-\tau) + \underline{b}(t), \tag{2.28}$$

where the coefficients A_0 and A_τ satisfy (2.1), *and $\underline{b} : \mathbb{R} \to \mathbb{R}^n$ is continuous. Consider also the ODE*

$$\dot{\hat{\underline{x}}}(t) = M\hat{\underline{x}}(t) + \hat{\underline{b}}(t), \tag{2.29}$$

where M is the solution of (2.3) *with property* (2.4), *and $\hat{\underline{b}} : \mathbb{R} \to \mathbb{R}^n$ is continuous. If $\hat{\underline{b}} : \mathbb{R} \to \mathbb{R}^n$ fulfills*

$$\hat{\underline{b}}(t) + A_\tau \int_{t-\tau}^{t} \mathrm{e}^{M(t-\tau-s)}\hat{\underline{b}}(s)\mathrm{d}s = \underline{b}(t) \quad \forall t \in \mathbb{R}, \tag{2.30}$$

then the following statements are valid:

1. *Every solution of the ODE* (2.29) *is a solution of the DDE* (2.28).

2. *For every solution $\underline{x} : [-\tau, \infty) \to \mathbb{R}^n$ of DDE* (2.28) *corresponding to some continuous initial function $\theta : [-\tau, 0] \to \mathbb{R}^n$, there exists a solution $\hat{\underline{x}}$ of ODE* (2.29) *such that*

$$\underline{x}(t) - \hat{\underline{x}}(t) \to 0 \qquad \text{as } t \to \infty. \tag{2.31}$$

Proof. First we show that if (2.30) holds, then

$$\hat{\underline{x}}_P(t) = \int_0^t \mathrm{e}^{M(t-s)}\hat{\underline{b}}(s)\mathrm{d}s, \qquad t \in \mathbb{R}, \tag{2.32}$$

is a common particular solution of equations (2.28) and (2.29).

According to the variation-of-constants formula, $\hat{\underline{x}}_P$ is a particular solution of ODE (2.29), that is,

$$\dot{\hat{\underline{x}}}_P(t) = M\hat{\underline{x}}_P(t) + \hat{\underline{b}}(t), \qquad t \in \mathbb{R}.$$

From this and the relation

$$\underline{\hat{x}}_P(t-\tau) = \int_0^{t-\tau} \mathrm{e}^{M(t-\tau-s)}\underline{\hat{b}}(s)\mathrm{d}s = \\ = \mathrm{e}^{-M\tau}\int_0^{t} \mathrm{e}^{M(t-s)}\underline{\hat{b}}(s)\mathrm{d}s - \mathrm{e}^{-M\tau}\int_{t-\tau}^{t} \mathrm{e}^{M(t-s)}\underline{\hat{b}}(s)\mathrm{d}s = \\ = \mathrm{e}^{-M\tau}\underline{\hat{x}}_P(t) - \mathrm{e}^{-M\tau}\int_{t-\tau}^{t} \mathrm{e}^{M(t-s)}\underline{\hat{b}}(s)\mathrm{d}s,$$

we find that

$$\dot{\underline{\hat{x}}}_P(t) - A_0\underline{\hat{x}}_P(t) - A_\tau\underline{\hat{x}}_P(t-\tau) = \underbrace{(M - A_0 - A_\tau\mathrm{e}^{-M\tau})}_{O_n}\underline{\hat{x}}_P(t) + \\ + \underline{\hat{b}}(t) + A_\tau\mathrm{e}^{-M\tau}\int_{t-\tau}^{t} \mathrm{e}^{M(t-s)}\underline{\hat{b}}(s)\underline{d}s = \\ = \underline{\hat{b}}(t) + A_\tau\int_{t-\tau}^{t} \mathrm{e}^{M(t-\tau-s)}\underline{\hat{b}}(s)\mathrm{d}s.$$

Therefore $\underline{\hat{x}}_P$ is a solution of (2.28) if and only if (2.30) holds.

It is known that every solution $\underline{\hat{x}}$ of ODE (2.29) has the form $\underline{\hat{x}} = \underline{\hat{x}}_H + \underline{\hat{x}}_P$, where $\underline{\hat{x}}_P$ is the particular solution from (2.32). By Theorem 2.3.12, $\underline{\hat{x}}_H$ is a solution of DDE (1.14) and, as shown before, $\underline{\hat{x}}_P$ is a solution of DDE (2.28). This implies that $\underline{\hat{x}} = \underline{\hat{x}}_H + \underline{\hat{x}}_P$ is a solution of DDE (2.28). Now let $\underline{x}$ be an arbitrary solution of DDE (2.28). Since both $\underline{x}$ and $\underline{\hat{x}}_P$ are solutions of DDE (2.28), we have that $\underline{x}_H = \underline{x} - \underline{\hat{x}}_P$ is a solution of the homogeneous equation (1.14). Theorem 2.3.12 guarantees the existence of a solution $\underline{\hat{x}}_H$ of equation (2.2) such that $\underline{x}_H(t) - \underline{\hat{x}}_H(t) \to 0$ as $t \to \infty$. If we define $\underline{\hat{x}}(t) = \underline{\hat{x}}_H(t) + \underline{\hat{x}}_P(t)$, then $\underline{\hat{x}}$ is a solution of (2.29) such that

$$\underline{x}(t) - \underline{\hat{x}}(t) = \underline{x}_H(t) + \underline{\hat{x}}_P(t) - (\underline{\hat{x}}_H(t) + \underline{\hat{x}}_P(t)) = \underline{x}_H(t) - \underline{\hat{x}}_H(t) \to 0$$

as $t \to 0$. □

In general, the integral equation (2.30) cannot be solved explicitly. Next it is shown that if the non-homogeneous term $\underline{b}(t)$ is bounded, then $\underline{\hat{b}}(t)$ can be computed using an iterative algorithm.

Theorem 2.4.2. *Suppose that* (2.1) *holds. Let* $\underline{b} : [-\tau, \infty) \to \mathbb{R}^n$ *be a continuous and bounded function. The equation* (2.30) *has a unique bounded solution* $\underline{\hat{b}} : \mathbb{R} \to \mathbb{R}^n$ *which can be computed by the method of successive approximations*

$$\underline{\hat{b}}(t) = \lim_{k\to\infty} \underline{\hat{b}}_k(t), \quad \forall t \in \mathbb{R}, \tag{2.33}$$

where $\underline{\hat{b}}_0(t) = \underline{b}(t)$, $\forall t \in \mathbb{R}$, *and* $\underline{\hat{b}}_{k+1}(t)$ *is given by*

$$\underline{\hat{b}}_{k+1}(t) = -A_\tau\int_{t-\tau}^{t} \mathrm{e}^{M(t-\tau-s)}\underline{\hat{b}}_k(s)\mathrm{d}s + \underline{b}(t), \qquad t \in \mathbb{R}, \qquad k = 0, 1, \ldots. \tag{2.34}$$

Furthermore, the convergence in (2.33) *is exponential.*

Proof. Let $\mathcal{B} = \mathcal{BC}(\mathbb{R}, \mathbb{R}^n)$ denote the Banach space of continuous and bounded functions on $\mathbb{R}$ with the supremum norm,

$$\|\underline{b}\|_{\mathcal{B}} = \sup_{t\in\mathbb{R}} \|\underline{b}(t)\|, \quad \underline{b} \in \mathcal{B}.$$

On $\mathcal{B}$ define an operator T by

$$(T\underline{\hat{b}})(t) = \underline{b}(t) - A_\tau \int_{t-\tau}^{t} \mathrm{e}^{M(t-\tau-s)} \underline{\hat{b}}(s)\mathrm{d}s$$

for $\underline{\hat{b}} \in \mathcal{B}$, $t \in \mathbb{R}$.

For $\underline{\hat{b}}_1, \underline{\hat{b}}_1 \in \mathcal{B}$, and $t \in \mathbb{R}$, we have

$$\begin{aligned} &\|(T\underline{\hat{b}}_1)(t) - (T\underline{\hat{b}}_2)(t)\| \\ &\qquad \leq \|A_\tau\| \int_{t-\tau}^{t} \mathrm{e}^{\|M\|(s-t-\tau)} \|\underline{\hat{b}}_2(s) - \underline{\hat{b}}_1(s)\|\mathrm{d}s \\ &\qquad \leq \|A_\tau\| \int_{t-\tau}^{t} \mathrm{e}^{\|M\|(s-t-\tau)}\mathrm{d}s \|\underline{\hat{b}}_2 - \underline{\hat{b}}_1\|_{\mathcal{B}} \\ &\qquad \leq \underbrace{\|A_\tau\|\mathrm{e}^{\nu_1\tau}\tau}_{\kappa} \|\underline{\hat{b}}_2 - \underline{\hat{b}}_1\|_{\mathcal{B}}. \end{aligned}$$

This shows that

$$\|(T\underline{\hat{b}}_1) - (T\underline{\hat{b}}_2)\|_{\mathcal{B}} \leq \kappa \|\underline{\hat{b}}_1 - \underline{\hat{b}}_2\|_{\mathcal{B}},$$

for $\underline{\hat{b}}_1, \underline{\hat{b}}_2 \in \mathcal{B}$. We have $0 < \kappa < 1$, see (2.12). Hence, $T : \mathcal{B} \to \mathcal{B}$ is a contraction and T has a unique fixed point $\underline{\hat{b}}$ in $\mathcal{B}$, which is a solution of (2.30) (Appendix B.1.2). The exponential convergence also follows from the Banach's theorem. □

Remark 2.4.3. The convergence of the iteration can also be directly analysed:

$$\|\underline{\hat{b}}(t) - \underline{\hat{b}}_{k+1}(t)\| \leq \|A_\tau\| \int_{t-\tau}^{t} \mathrm{e}^{\|M\|(\tau+s-t)} \|\underline{\hat{b}}(s) - \underline{\hat{b}}_{k+1}(s)\|\mathrm{d}s, \qquad t \in \mathbb{R}, k = 0, 1, \dots,$$

from which we have

$$\|\underline{\hat{b}} - \underline{\hat{b}}_{k+1}\|_{\mathcal{B}} \leq \kappa \|\underline{\hat{b}} - \underline{\hat{b}}_k\|_{\mathcal{B}}, \qquad k = 0, 1, \dots,$$

similarly as before. Now the convergence rate can be given by as

$$\|\underline{\hat{b}} - \underline{\hat{b}}_k\|_{\mathcal{B}} \leq \kappa^{k-1} \|\underline{\hat{b}} - \underline{b}\|_{\mathcal{B}}, \qquad k = 1, 2, \dots,$$

which shows that the values $\underline{\hat{b}}_k(t)$ converge to $\underline{\hat{b}}(t)$ uniformly on $\mathbb{R}$ as $k \to \infty$ at an exponential rate.

Remark 2.4.4. In the case of a constant non-homogeneous term $\underline{b}(t) \equiv \underline{b}$ the unique bounded solution of is the constant vector (2.30) is the constant vector

$$\underline{\hat{b}} = \left(I + A_\tau \int_0^\tau \mathrm{e}^{-sM}\mathrm{d}s \right)^{-1} \underline{b}. \tag{2.35}$$

2.5 Application to observer design

In this section, results of the previous sections are used to give a simple observer for a continuous-time linear TDS with discrete delay.

2.5.1 Observability and detectability

In system theory, the observability of a system is a characteristic, which shows whether the internal states can be inferred from the external outputs.

Consider the TDS extended with linear output mapping

$$\begin{cases} \underline{\dot{x}}(t) = A_0\underline{x}(t) + A_\tau \underline{x}(t-\tau) + \underline{b}(t) \\ \underline{y}(t) = C\underline{x}(t) \end{cases}, \tag{2.36}$$

with a constant initial function $\underline{x}(t) = \underline{\phi} \in \mathbb{R}^n$ for $t \in [-\tau, 0]$, $\underline{y}(t) \in \mathbb{R}^p$ output vector and $C \in \mathbb{R}^{p\times n}$ output matrix.

Definition 2.5.1. [54] Let $t_0, t_1 \in \mathbb{R}$, such that $t_0 \le t_1 < \infty$. The system (2.36) is observable on $[t_0, t_1]$ if for all initial functions, the initial vector $\underline{x}(t_0)$ can be uniquely determined from the initial function and from the output function $\underline{y}(t)$ on $[t_0, t_1]$.

Theorem 2.5.1. *[54] The system* (2.36) *is observable on* $[0, t]$, $\forall t > n\tau$ *if* $\operatorname{rank}(P) = n$, *where*

$$P = [P_1^1, \dots P_1^n, P_2^2, \dots, P_2^n, \dots, P_n^n]C^\top,$$

with

$$P_1^1 = I_n, \tag{2.37}$$

$$P_j^{k+1} = A_0^\top P_j^k + A_\tau^\top P_{j-1}^k, \quad j = 1, \dots k+1, \quad k = 1, \dots n-1, \tag{2.38}$$

and $P_j^k = O_n$*, if* $j = 0$*, or* $j > k$*.*

Example 2.5.1. In case of a system with states $\underline{x} \in \mathbb{R}^2$ and appropriate matrices, the observability matrix is $P = [P_1^1, P_1^2, P_2^2]C^\top$, where $P_1^1 = I$, $P_1^2 = A_0^\top P_1^1 + A_\tau^\top P_0^1 = A_0^\top$ and $P_2^2 = A_0^\top P_2^1 + A_\tau^\top P_1^1 = A_\tau^\top$. Finally the observability matrix can be written as

$$P = [I, A_0^\top, A_\tau^\top]C^\top.$$

In case of a system with states $\underline{x} \in \mathbb{R}^3$ and appropriate matrices, the observability matrix is written in the form

$P = [P_1^1, P_1^2, P_2^3, P_2^2, P_2^3, P_3^3]C^\top$.
$P_1^1 = I$,
$P_1^2 = A_0^\top P_1^1 + A_\tau^\top P_0^1 = A_0^\top$,
$P_1^3 = A_0^\top P_1^2 + A_\tau^\top P_0^2 = (A_0^2)^\top$,
$P_2^2 = A_0^\top P_2^1 + A_\tau^\top P_1^1 = A_\tau^\top$,
$P_2^3 = A_0^\top P_2^2 + A_\tau^\top$,
$P_1^2 = A_0^T A_\tau^\top + A_\tau^\top A_0^\top$ and
$P_3^3 = A_0^\top P_3^2 + A_\tau^\top P_2^2 = (A_\tau^2)^\top$.

The observability matrix is written as

$$P = [I, A_0^\top, (A_0^2)^\top, A_\tau^\top, A_0^\top P_2^2 + A_\tau^\top, (A_\tau^2)^\top]C^\top$$

.

Theorem 2.5.1 shows that the delay value does not influence the observability of a system.

If the rank condition $\operatorname{rank}(P) = n$ is not satisfied, then there are unobservable entries in the state vector $\underline{x}(t)$.

Definition 2.5.2. [67] The system (2.36) is called detectable if all unobservable state vector entries are stable.

Theorem 2.5.2. *[68, Theorem 4.2] The system* (2.36) *is detectable if and only if*

$$\operatorname{rank}\left(C, \lambda I_n - A_0 - A_\tau e^{-\lambda\tau}\right) = n, \quad \forall \lambda \in \mathbb{C} : \Re(\lambda) \geq 0. \tag{2.39}$$

Detectability conditions are essential, e.g. as shown by Li et al. [69] in the case of sensor networks. In large-scale sensor networks, the communication delay can not be neglected. It is why it is essential to treat the detectability in the presence of delay. Next, a simplified detectability condition is given for TDSs, which is a consequence of Theorems 2.3.9 and 2.5.2.

Theorem 2.5.3. *Consider the system* (2.36) *which satisfies* (2.1). *The system is detectable if*

$$\operatorname{rank}\left(C, \lambda I_n - M\right) = n, \quad \forall \lambda \in \mathbb{C} : \Re(\lambda) \geq 0, \tag{2.40}$$

where M is the solution of (2.3) *with property* (2.4).

2.5.2 Observer design

Now the implementation steps of an observer are presented for a continuous-time linear TDS which satisfy (2.1) and compare it to two different existing methods: observer design based on Galerkin's approximation with tau incorporation shown by Chakraborty et al. [70], and the continuous pole placement method developed by Michiels et al. [71].

Consider a TDS in the form of (2.36), which is observable and it satisfies (2.1). The following algorithm will create a delay-free observer system, where the states converge exponentially to the states of the original, delayed system:

Algorithm1

- Compute M given by (2.3) satisfying (2.4).
- Compute $\hat{\underline{b}}(t)$ given by (2.30).
- Give an observer gain matrix K such that the homogeneous part of the observer system

$$\dot{\underline{\tilde{x}}}(t) = M\underline{\tilde{x}}(t) + \hat{\underline{b}}(t) + K(\underline{y}(t) - C\underline{\tilde{x}}(t)), \tag{2.41}$$

  is asymptotically stable i.e. $M - KC$ is Hurwitz. Here $\underline{y}$ is the output of the observed system (2.36).

Proposition 2.5.4. *Consider an observable TDS given by* (2.36) *such that* (2.1) *holds, and $\underline{b} : \mathbb{R} \to \mathbb{R}^n$ is continuous and bounded. Then the state observer* (2.41) *designed using Algorithm 1 assures that*

$$\lim_{t\to\infty} \|\underline{x}(t) - \underline{\tilde{x}}(t)\| = 0,$$

where $\underline{x}$ and $\underline{\tilde{x}}$ are arbitrary solutions of (2.36) *and* (2.41), *respectively.*

Proof. Let $\underline{x}$ and $\underline{\tilde{x}}$ arbitrary solutions of (2.36) and (2.41), respectively. By Theorem 2.4.1, there exists a solution $\underline{\hat{x}}$ of (2.29) such that (2.31) holds. Define

$$\underline{\epsilon}(t) = \underline{\hat{x}}(t) - \underline{\tilde{x}}(t) \quad \text{for } t \in \mathbb{R}.$$

From (2.29), (2.36) and (2.41), we obtain for $t \geq 0$,

$$\dot{\underline{\epsilon}}(t) = (M - KC)\underline{\epsilon}(t) + \underline{f}(t),$$

where

$$\underline{f}(t) = KC(\hat{\underline{x}}(t) - \underline{x}(t)) \quad \text{for } t \geq 0.$$

Note that $M - KC$ is Hurwitz and (2.31) implies that $\underline{f}(t) \to \underline{0}$ as $t \to \infty$. By the application of [72, Chap. III, Theorem 8], we conclude that

$$\underline{\epsilon}(t) = \hat{\underline{x}}(t) - \tilde{\underline{x}}(t) \to \underline{0} \quad \text{as } t \to \infty.$$

This, together with (2.31), implies that

$$\underline{x}(t) - \tilde{\underline{x}}(t) = (\underline{x}(t) - \hat{\underline{x}}(t)) + (\hat{\underline{x}}(t) - \tilde{\underline{x}}(t)) \to \underline{0} \quad \text{as } t \to \infty.$$

□

In Garlekin's approximation based design, the state vector must be extended, and the observer gain can be obtained using symbolic computation. The continuous pole placement method is an iterative algorithm, which requires eigenvalue computation in each iteration step. In smallness condition-based observer design, the observer gain can be obtained using standard linear observer design methods, such as the pole placement method or linear quadratic estimator design.

2.6 Case studies

In this section, examples are provided for the proposed observer design. The method was compared with the observer design based on Galerkin's approximation method with tau incorporation using 5^{th} order Legendre polynomials (for the complete application, see Appendix A).

Example1:

Consider a system with $A_0 = \begin{pmatrix} -0.6082 & 0.3159 \\ 0.0887 & -0.8846 \end{pmatrix}$, $A_\tau = \begin{pmatrix} -0.7429 & 0.3947 \\ 0.2815 & 0.5485 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 1 \end{pmatrix}$. Let the non-homogeneous term be $\underline{b}(t) = B\underline{u}(t)$, with $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The time delay is $\tau = 0.27s$, and the initial function is $\underline{x}(t) = \underline{1}$, $\forall t \in [-\tau, 0]$.

First the smallness condition (2.1) is checked, which is $0.8089 < 1$, so that M and $\hat{\underline{b}}$ can be computed. Next the observability of the system is checked based on Example 2.5.1: $P = \begin{pmatrix} 0 & 0.0887 & 0.2815 \\ 1 & -0.8846 & 0.5485 \end{pmatrix}$, $\text{rank}(P) = 2$, which means the system is observable for all $t > 2\tau = 0.54s$.

The approximate system matrix is $M = \begin{pmatrix} -1.96 & 1.0567 \\ 0.4762 & -0.3761 \end{pmatrix}$ after the 10^{th} iteration using the recursive formula (2.6).

$\hat{\underline{b}} = \hat{B}\underline{u}(t)$ is calculated, presuming that $\underline{u}(t)$ is piecewise linear, using relation (2.35) as $\hat{B} = \begin{pmatrix} 1.415 \\ 0 \end{pmatrix}$. The observer gain is chosen such that the eigenvalues of $M - KC$ are $\lambda_1 = -2.5$, $\lambda_2 = -20$, which yields $K = \begin{pmatrix} 18.45 \\ 19.7 \end{pmatrix}$.

On Figure 2.2 the state strajectories are shown, while Figure 2.3 shows the relative error calculated as $\underline{e}(t) = 100\frac{\|\underline{x}(t) - \hat{\underline{x}}(t)\|}{\underline{x}(t)}$.

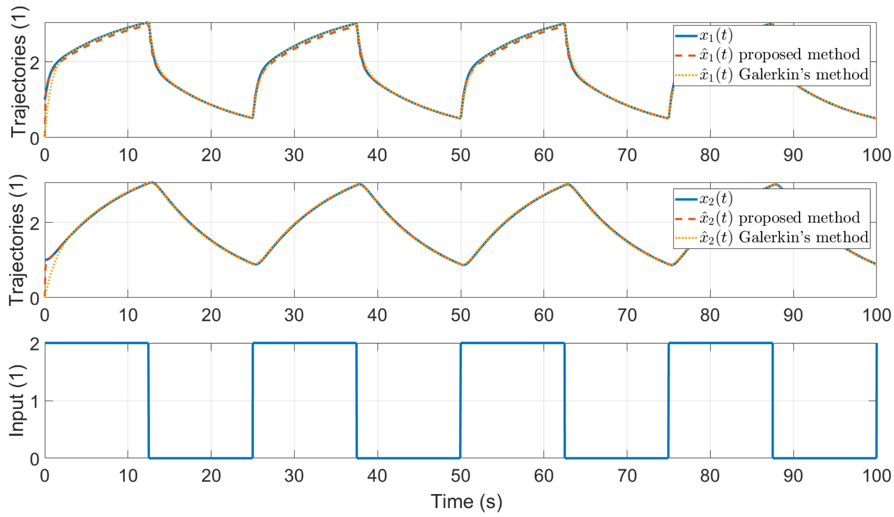

Figure 2.2 Trajectories of the TDS and the approximating systems.

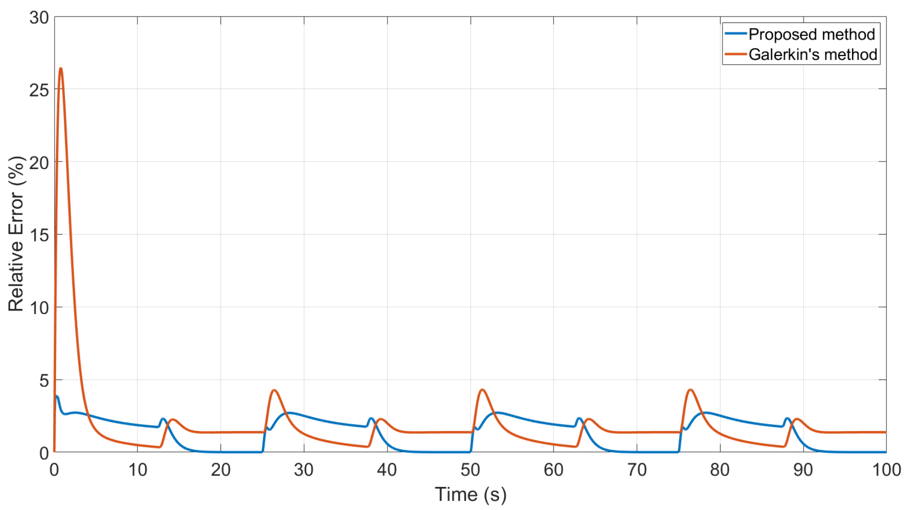

Figure 2.3 The relative approximation error of the observer for the example system 2.6.

Example2:

Consider an unstable system with $A_0 = O_2$, $A_\tau = \begin{pmatrix} 0 & 0.3 \\ 0.3 & 0. \end{pmatrix}$, $C = \begin{pmatrix} 0 & 1 \end{pmatrix}$. Let the non-homogeneous term be $\underline{b}(t) = B\underline{u}(t)$, with $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The time delay is $\tau = 1s$, and the initial function is $\underline{x}(t) = \underline{1}$, $\forall t \in [-\tau, 0]$.

First the smallness condition (2.1) is checked, which is $0.8155 < 1$, so that M and $\underline{\hat{b}}$ can be computed. Next the observability of the system is checked based on Example 2.5.1: $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0.3 \end{pmatrix}$, $\text{rank}(P) = 2$, which means the system is observable $\forall t > 2\tau = 2s$.

The approximate system matrix is $M = \begin{pmatrix} -0.1236 & 0.3631 \\ 0.3631 & -0.1263 \end{pmatrix}$ after the 10^{th} iteration using the recursive formula (2.6).

$\hat{\underline{b}} = \hat{B}\underline{u}(t)$ is calculated, presuming that $\underline{u}(t)$ is piecewise linear, using relation (2.35) as $\hat{B} = \begin{pmatrix} 1.16 \\ 0.3517 \end{pmatrix}$. The observer gain is chosen such that the eigenvalues of $M - KC$ are $\lambda_1 = -0.12$, $\lambda_2 = -12$, which yields $K = \begin{pmatrix} 0.5 \\ 11.7 \end{pmatrix}$.

Figure 2.4 shows the relative error.

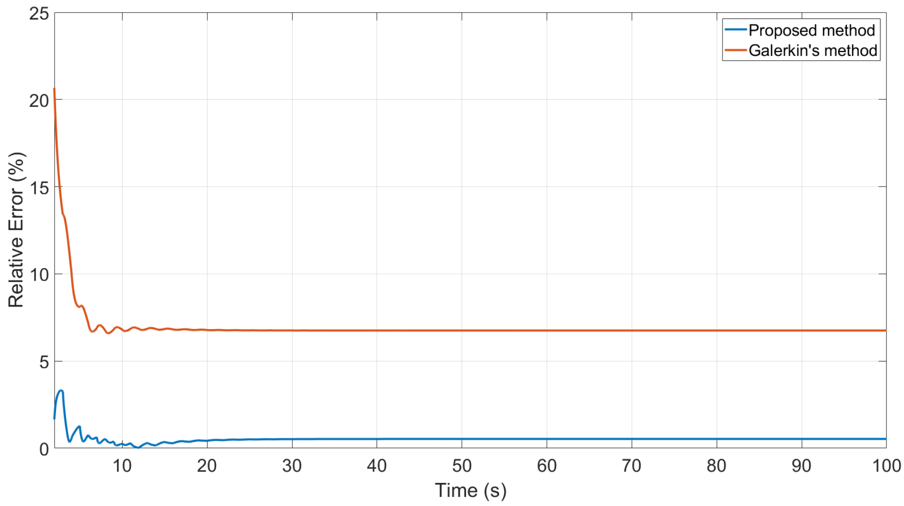

Figure 2.4 The relative approximation error of the observer for the example system 2.6.

Example3:
Consider the system (2.36) with

$$A_0 = \begin{pmatrix} -0.386 & 0 & 0 \\ 0 & -0.193 & 0 \\ 0 & 0 & -0.386 \end{pmatrix},$$

$$A_\tau = \begin{pmatrix} 0 & 0 & 0 \\ 0.125 & 0 & 0.125 \\ 0 & 0.25 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Let the non-homogeneous term be $\underline{b}(t) = B\underline{u}(t)$, with

$$B = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

The time delay is set $\tau = 1s$, and the initial function is

$$\underline{x}(t) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \forall t \in [-\tau, 0]$$

.

First the smallness condition (2.1) is checked, which is $0.9997 < 1$, so M and $\hat{\underline{b}}$ can be computed.

Next the observability of the system is checked based on Example 2.5.1:

$$P = \begin{pmatrix} 0.005 & 0.0063 & 0.0055 \\ 0.0063 & 0.0084 & 0.072 \\ 0.0055 & 0.0072 & 0.0062 \end{pmatrix},$$

$\text{rank}(P) = 3$, which means the system is observable $\forall t > 3\tau = 3s$.

The state matrix of the approximate system is

$$M = \begin{pmatrix} -0.3860 & 0 & 0 \\ 0.2043 & -0.2534 & 0.2043 \\ -0.0739 & 0.3340 & -0.4599 \end{pmatrix}$$

after the 10^{th} iteration using the recursive formula (2.6).

$\hat{\underline{b}} = \hat{B}\underline{u}(t)$ is calculated, presuming that $\underline{u}(t)$ is piecewise linear as it is shown in Figure 2.5, using relation (2.35) as

$$\hat{B} = \begin{pmatrix} 0.4504 \\ 0.5207 \\ 0.4737 \end{pmatrix}.$$

The observer gain was computed using the Matlab *place* function such that the eigenvalues of $M - KC$ are $\lambda_1 = -0.6$, $\lambda_2 = -3.5$, $\lambda_3 = -13.5$.

$$K = \begin{pmatrix} 12.3 & 1.18 \\ 0.63 & 1.56 \\ 5.71 & 2.59 \end{pmatrix}.$$

In Figure 2.5 the state strajectories are shown, while Figure 2.6 shows the relative error calculated as $\underline{e}(t) = \frac{\|\underline{x}(t)-\hat{\underline{x}}(t)\|}{\|\underline{x}(t)\|}100(\%)$. The average relative error in the case of Garlekin's method is 5.12%, while the proposed method is 3.96%.

2.7 Summary

The presence of time delay makes the analysis of dynamic systems difficult, even in the linear case. In this chapter, a constructive approximation procedure is proposed for a class of continuous linear TDS with additive non-homogeneous terms. An iterative method was given to compute the delay-free approximate system. The resulting approximate system has the same state dimension as the original delay system, and the trajectories converge exponentially to the trajectories of the original TDS.

Based on the proposed approximation, it was shown that the observer design for the addressed class of TDSs could be traced back to the observer design for delay-free systems. A delay-free detectability condition was also developed for small gain TDSs.

The numerical evaluation of the proposed algorithm shows that the proposed observer can estimate the states of the time delay systems with small gains efficiently.

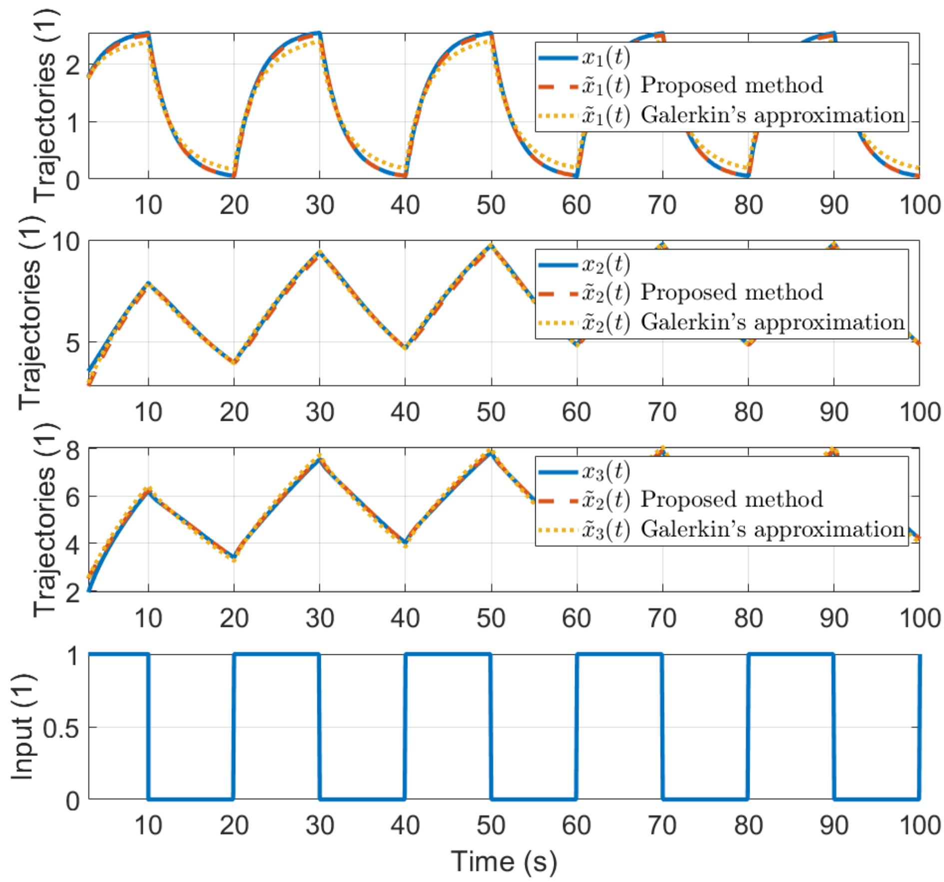

Figure 2.5 Trajectories of the TDS and the approximating systems.

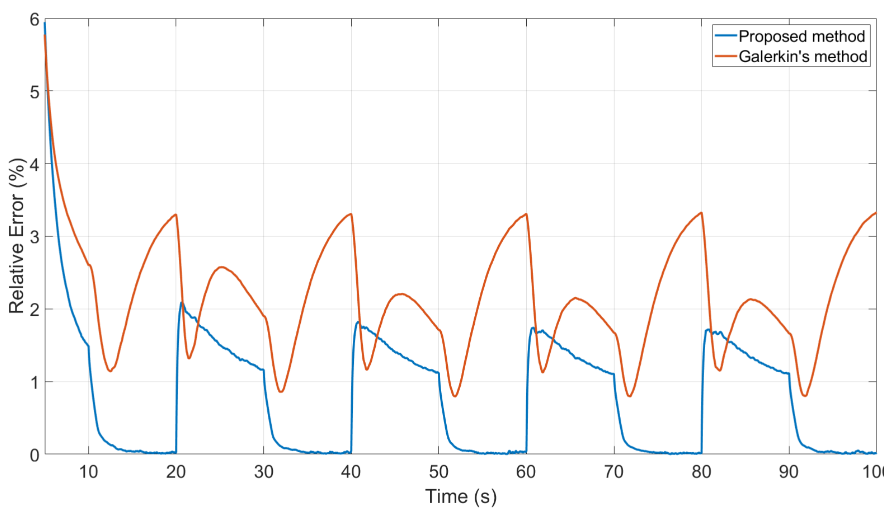

Figure 2.6 The relative approximation error of the observer for the example system 2.6.

Chapter 3

Approximation of discrete time Volterra type linear systems with infinite delays

3.1 Abstract

In this chapter, linear Volterra difference equations with infinite delays are considered. It is shown that if the coefficient matrices are sufficiently small, then the Volterra difference equation is asymptotically equivalent to a linear ordinary difference equation at infinity. An efficient new method is obtained to compute the characteristic roots and the system matrix of the ordinary difference equation with explicit error estimates. Furthermore, it is shown that the approximation method can be used to analyse multi-agent systems in the presence of communication delays. Simulation results are given to support the applicability of the presented method. This chapter is based on [73*, 74*].

3.2 Literaure survey

The system (1.18) arises as a model for evolutionary processes which takes history into account. Volterra difference equations may be viewed as numerical approximations of Volterra integral and integrodifferential equations, which have critical applications in population dynamics as shown by Brunner et al. [75], Cushing [76], and Lubich [77]. Discrete population models described by Volterra difference equations can be found in Chap. 5 of the recent monograph by Raffoul [78]. It should be noted that Volterra difference equations contain as a special case ordinary difference equations and difference equations with finite delay. For applications of difference equations in biology, economics and engineering, refer to the monograph by Elaydi [79] and the references therein.

A MAS is a system composed of multiple interacting agents, which can be used to solve complex tasks that can not be accomplished by an individual or monolithic agent [80]. Such systems are widely used to model interconnected power systems [81], sensor networks [82, 83], and robotic swarms [84, 85].

In MAS, the consensus is a process by which multiple agents reach an agreement on the value of a variable using locally available information [86]. Besides the steady-state consensus value, the analysis of transient behaviour is also crucial in many applications. Mosebach et al. developed a consensus algorithm for asymptotic synchronisation for both continuous and discrete-time MAS which fulfilled prescribed transient performances as synchronisation time and minimum damping criteria [87].

By increasing the number of agents, the inter-agent distance, or the complexity of the task in a MAS, the effects of the delays cannot be ignored, as shown in [88]. Such delays are the input delay [89] which is caused by the processing time of the individual agent, and the communication delay [90].

The effect of the time-delay in MAS was studied in multiple articles. Wang et al. [91] gave sufficient conditions for the consensus of the first-order MAS with unstable agents over an undirected network in the presence of constant and time-varying communication delay. Zheng et al. [92] found a condition on the delay, packet drop-out rates, the communication topology and agent dynamics, in case of a discrete MAS with packet drop-out and communication delay, under which there exists a control such that the MAS is mean-square consensusable. Liang et al. [93] studied the convergence problem of the first-order MAS in the presence of multiple communication, and input delays based on the generalised Nyquist criterion and frequency-domain analysis. Zhao et al. [94] showed that the average consensus of discrete MAS is robust in the presence of uniform, constant time-delay, but the steady-state drifts compared to the system, which does not contain delays. Explicit expression was given for this drift in.

The analysis of linear discrete-time delay system is usually done by creating the augmented state matrix [54]. In case of a system with n states and a discrete-time, constant delay q this results in an augmented system with $n \times q$ states.

3.3 Approximation of the homogeneous Volterra difference equation

Consider the Volterra difference equation (1.18). Throughouth the chapter, it is assumed that

$$\sum_{j=0}^{\infty} \|A[j]\| \nu_0^{-j} < 1 - \nu_0 \quad \text{for some } \nu_0 \in (0,1). \tag{3.1}$$

Equation (1.18) may be viewed as a special case of a linear functional difference equation with phase space

$$\mathcal{B}_{\nu_0} = \{\underline{\phi} : \mathbb{Z}_- \to \mathbb{R}^n \mid \sup_{j \in \mathbb{Z}_-} \|\underline{\phi}[j]\| \nu_0^{-j} < \infty\} \tag{3.2}$$

equipped with the norm

$$\|\underline{\phi}\|_{\mathcal{B}_{\nu_0}} = \sup_{j \in \mathbb{Z}_-} \|\underline{\phi}[j]\| \nu_0^{-j}, \quad \text{for } \phi \in \mathcal{B}_{\nu_0} \tag{3.3}$$

(see [95]). If the infinite series in (3.1) is convergent, then for every $\underline{\phi} \in \mathcal{B}_{\nu_0}$, equation (1.18) has a unique solution $\underline{x} : \mathbb{Z} \to \mathbb{R}^n$ satisfying (1.18) for all $k \in \mathbb{Z}_+$ with initial data

$$\underline{x}[j] = \underline{\phi}[j] \quad \text{for } j \in \mathbb{Z}_-. \tag{3.4}$$

Assumption (3.1) implies that the z-transform $A^*(z) = \sum_{j=0}^{\infty} A[j] z^{-j}$ converges at $z = \nu_0$ and hence $r \le \nu_0 < 1$, where r is the radius of convergence of A^* given by

$$r = \limsup_{j \to \infty} \sqrt[j]{\|A[j]\|}. \tag{3.5}$$

This shows that if (3.1) holds, then $\|A[j]\| \to 0$ exponentially as $j \to \infty$. Since each $\nu_0 \in (0,1)$ can be written in the form $\nu_0 = \mu(\mu+1)^{-1}$ for some $\mu > 0$, condition (3.1) is equivalent to

$$\sum_{j=0}^{\infty} \|A[j]\| \frac{(\mu+1)^{j+1}}{\mu^j} < 1 \quad \text{for some } \mu > 0. \tag{3.6}$$

Thus, assumption (3.1)may be viewed as a *smallness condition* on the coefficients $A[j]$, $j \in \mathbb{N}$.

In this chapter, it will be shown that under the smallness condition (3.1) the Volterra difference equation (1.18) is asymptotically equivalent to the ordinary difference equation

$$\underline{\hat{x}}[k+1] = M\underline{\hat{x}}[k], \tag{3.7}$$

where $M \in \mathbb{R}^{n\times n}$ is an appropriate (invertible) solution of the matrix equation

$$M = I_n + \sum_{j=0}^{\infty} A[j]M^{-j}. \tag{3.8}$$

As corollary, an asymptotic description of the solutions of the Volterra difference equation (1.18) is obtained in terms of the eigensolutions of the ordinary difference equation (3.7). In addition, it will be shown that the eigenvalues of M coincide with the roots of the characteristic equation (1.19). Moreover, M can be written as a limit of successive approximations

$$M = \lim_{l\to\infty} M_l \tag{3.9}$$

with

$$M_0 = I_n \quad \text{and} \quad M_{l+1} = I_n + \sum_{j=0}^{\infty} A[j]M_l^{-j} \quad \text{for } l = 0,1,\ldots, \tag{3.10}$$

where the convergence in (3.9) is exponential. As a consequence, it is shown that the eigenvalues of the successive approximations M_l converge to the roots of the characteristic equation (1.19) at an exponential rate as $l \to \infty$. This yields an efficient new method for the approximation of the characteristic roots of (1.18).

3.3.1 Solution of the associated matrix equation

First some properties of the roots of the scalar equation

$$\sum_{j=0}^{\infty} \|A[j]\|\nu^{-j} = 1-\nu \tag{3.11}$$

are proven.

Lemma 3.3.1. *If* (3.1) *holds, then* (3.11) *has a unique root* $\nu_1 \in (\nu_0, 1)$. *Moreover,*

$$\kappa = \sum_{j=1}^{\infty} j\|A[j]\|\nu_1^{-j-1} < 1. \tag{3.12}$$

Proof. The roots of (3.11) coincide with the roots of the function

$$f(\nu) = 1-\nu-\sum_{j=0}^{\infty} \|A[j]\|\nu^{-j} \tag{3.13}$$

defined for $\nu \geq \nu_0$. For $\nu > \nu_0$, we have

$$f'(\nu) = -1 + \sum_{j=0}^{\infty} j\|A[j]\|\nu^{-j-1} \tag{3.14}$$

and

$$f''(\nu) = -\sum_{j=0}^{\infty} j(j+1)\|A[j]\|\nu^{-j-2} < 0 \tag{3.15}$$

and hence $f'(\nu)$ is strictly decreasing on (ν_0, ∞). Two cases may occur: either $f'(\nu)$ has no root in (ν_0, ∞), or $f'(\nu_2) = 0$ for some $\nu_2 \in (\nu_0, \infty)$.

In the first case, $f'(\infty) = -1$ implies that $f'(\nu) < 0$ on (ν_0, ∞) and hence $f(\nu)$ is stricly decreasing on $[\nu_0, \infty)$. By virtue of (3.1) and (3.13), we have that $f(\nu_0) > 0$ and $f(1) < 0$ which implies the existence of $\nu_1 \in (\nu_0, 1)$ such that $f(\nu_1) = 0$. Since $f'(\nu) < 0$ on (ν_0, ∞), we have that $f'(\nu_1) < 0$ which implies (3.12). The uniqueness of ν_1 follows from the strict monotonicity of $f(\nu)$ on $[\nu_0, \infty)$.

In the second case, since $f'(\nu)$ is strictly decreasing on (ν_0, ∞), we have that $f'(\nu) > 0$ on (ν_0, ν_2) and $f'(\nu) < 0$ on (ν_2, ∞). As a consequence, $f(\nu)$ is strictly increasing on $[\nu_0, \nu_2]$ and $f(\nu)$ is strictly decreasing on $[\nu_2, \infty)$. We claim that $\nu_2 \in (\nu_0, 1)$. Otherwise, the strictly increasing property of $f(\nu)$ on $[\nu_0, \nu_2]$, together with (3.1), would imply that $0 < f(\nu_0) < f(1)$ contradicting the fact that $f(1) < 0$. Thus, we have that $\nu_2 \in (\nu_0, 1)$, $0 < f(\nu_0) < f(\nu_2)$ and $f(1) < 0$ which implies the existence of $\nu_1 \in (\nu_2, 1)$ such that $f(\nu_1) = 0$. Since $f'(\nu) < 0$ on (ν_2, ∞), we have that $f'(\nu_1) < 0$ and hence (3.12) holds. The uniqueness of ν_1 follows from the fact $f(\nu) > 0$ on $[\nu_0, \nu_2]$ and $f(\nu)$ is strictly decreasing on $[\nu_2, \infty)$. □

The following lemmas regarding the distances of the powers of inverse matrices are also needed.

Lemma 3.3.2. *Suppose that $P, Q \in \mathbb{R}^{n\times n}$ are invertible and let $\gamma = \max\{\|P^{-1}\|, \|Q^{-1}\|\}$. Then for every $j \in \mathbb{N}^*$,*

$$\|P^{-j} - Q^{-j}\| \leq j\gamma^{j+1}\|P - Q\|. \tag{3.16}$$

Proof. By Lemma 2.3.1, we have for $j \geq 1$,

$$\|P^{-j} - Q^{-j}\| = \|(P^{-1})^j - (Q^{-1})^j\| \leq j\gamma^{j-1}\|P^{-1} - Q^{-1}\|. \tag{3.17}$$

From the identity

$$P^{-1} - Q^{-1} = P^{-1}(Q - P)Q^{-1},$$

we find that

$$\|P^{-1} - Q^{-1}\| \leq \|P^{-1}\|\|Q - P\|\|Q^{-1}\| \leq \gamma^2\|P - Q\|.$$

The last inequality together with (3.17), implies (3.16). □

Lemma 3.3.3. *[96, Example 4.5] Let $P \in \mathbb{R}^{n\times n}$. If $\|P\| < 1$, then $I_n - P$ is invertible and its inverse is given by the Neumann series*

$$(I_n - P)^{-1} = \sum_{j=0}^{\infty} P^j. \tag{3.18}$$

Moreover,

$$\|(I_n - P)^{-1}\| \leq \frac{1}{1 - \|P\|}. \tag{3.19}$$

In the following theorem, the existence and uniqueness of the solution of the matrix equation (3.8) is proven.

Theorem 3.3.4. *Suppose* (3.1) *holds. Then the matrix equation* (3.8) *has a unique solution* $M \in \mathbb{R}^{n\times n}$ *such that*

$$\|I_n - M\| \leq 1 - \nu_1, \tag{3.20}$$

where ν_1 *is the unique root of* (3.11) *in the interval* $(\nu_0, 1)$. *Moreover,*

$$\|M^{-1}\| \leq \frac{1}{\nu_1} \tag{3.21}$$

and

$$|\lambda| \geq \nu_1 > \nu_0 \quad \textit{whenever } \lambda \in \sigma(M). \tag{3.22}$$

Proof. By Lemma 3.3.1, Equation (3.11) has a unique root $\nu_1 \in (\nu_0, 1)$. Let

$$\mathcal{S} = \{M \in \mathbb{R}^{n\times n} \mid \|I_n - M\| \leq 1 - \nu_1\}. \tag{3.23}$$

Evidently, $\mathcal{S}$ is a nonempty and closed subset of $\mathbb{R}^{n\times n}$. If $M \in \mathcal{S}$ then

$$\|I_n - M\| \leq 1 - \nu_1 < 1 \tag{3.24}$$

and therefore Lemma 3.3.3 implies that $M = I_n - (I_n - M)$ is invertible and

$$\|M^{-1}\| \leq \frac{1}{1 - \|I_n - M\|} \leq \frac{1}{\nu_1} \quad \text{whenever } M \in \mathcal{S}. \tag{3.25}$$

Define

$$F(M) = I_n + \sum_{j=0}^{\infty} A[j]M^{-j} \quad \text{for } M \in \mathcal{S}. \tag{3.26}$$

By virtue of (3.25) and (3.26), we have for $M \in \mathcal{S}$,

$$\|I_n - F(M)\| \leq \sum_{j=0}^{\infty} \|A[j]\|\|M^{-j}\| \leq \sum_{j=0}^{\infty} \|A[j]\|\|M^{-1}\|^j \leq \sum_{j=0}^{\infty} \|A[j]\|\nu_1^{-j} = 1 - \nu_1.$$

Thus, F maps $\mathcal{S}$ into itself.

Let $M_1, M_2 \in \mathcal{S}$. From relation (3.25), we have that

$$\gamma = \max\{\|M_1^{-1}\|, \|M_2^{-1}\|\} \leq \frac{1}{\nu_1}.$$

From this and (3.24), by the application of Lemma 3.3.2, we find that

$$\|F(M_1) - F(M_2)\| \leq \sum_{j=1}^{\infty} \|A[j]\|\|M_1^{-j} - M_2^{-j}\| \leq \underbrace{\sum_{j=1}^{\infty} \|A[j]\|j\nu_1^{-j-1}}_{\kappa} \|M_1 - M_2\|$$

with κ as in (3.12). Since $\kappa < 1$, $F : \mathcal{S} \to \mathcal{S}$ is a contraction. By Banach's theorem (Appendix B.1.2), F has a unique fixed point $M \in \mathcal{S}$ which is the unique solution of the matrix equation (3.8) satisfying (3.20). Note that (3.21) is a consequence of (3.25). It remains to prove (3.22). If $\lambda \in \sigma(M)$ is an eigenvalue of M and $\underline{u}$ is a corresponding eigenvector, then $M^{-1}\underline{u} = \lambda^{-1}\underline{u}$ which implies that $\lambda^{-1} \in \sigma(M^{-1})$

and hence

$$|\lambda|^{-1} = |\lambda^{-1}| \leq \rho(M^{-1}) \leq \|M^{-1}\| \leq \nu_1^{-1},$$

where the last inequality is a consequence of (3.21). This implies (3.22). □

Next it is shown that the unique solution of the matrix equation (3.8) satisfying (3.20) can be written as a limit of successive approximations M_l defined by (3.10) and an estimate is given for the approximation error.

Theorem 3.3.5. *Suppose* (3.1) *holds and let* $M \in \mathbb{R}^{n\times n}$ *be the solution of* (3.8) *satisfying* (3.20). *If* $\{M_l\}_{l=0}^{\infty}$ *is the sequence of matrices defined by* (3.10), *then*

$$\|I_n - M_l\| \leq 1 - \nu_1 \quad \text{for } l = 0, 1, \ldots, \tag{3.27}$$

and

$$\|M - M_l\| \leq (1-\nu_1)\kappa^l \quad \text{for } l = 0, 1, \ldots, \tag{3.28}$$

where ν_1 *is the unique root of* (3.11) *in* $(\nu_0, 1)$ *and* $\kappa < 1$ *is given by* (3.12).

Proof. Referring to the proof of Theorem 3.3.4, we have that $M_0 = I_n \in \mathcal{S}$ and therefore $F(\mathcal{S}) \subset \mathcal{S}$ implies by easy induction on l that $M_{l+1} = F(M_l) \in \mathcal{S}$ for all $l = 0, 1, 2, \ldots$. Thus, (3.27) holds. Moreover, by virtue of (3.25), we have

$$\|M_l^{-1}\| \leq \frac{1}{\nu_1} \quad \text{for } l = 0, 1, 2, \ldots. \tag{3.29}$$

From this and (3.25), we find for each $l \geq 1$,

$$\max\{\|M^{-1}\|, \|M_l^{-1}\|\} \leq \frac{1}{\nu_1}. \tag{3.30}$$

From this, by the application of Lemma 3.3.2, we obtain for $l \in \mathbb{N}$,

$$\|M - M_{l+1}\| = \|F(M) - F(M_l)\| \leq \sum_{j=1}^{\infty} \|A[j]\| \|M^{-j} - M_l^{-j}\| \leq$$
$$\leq \sum_{j=1}^{\infty} \|A[j]\| j\nu_1^{-j-1} \|M - M_l\|.$$

From the last inequality, we obtain by induction on l,

$$\|M - M_l\| \leq \kappa^l \|M - M_0\| = \kappa^l \|M - I_n\| \quad \text{for } l = 0, 1, \ldots.$$

This, together with (3.20), implies (3.28). □

3.3.2 Approximation of characteristic roots

In this section, it will be shown that the characteristic roots of Equation (1.18) can be approximated by the eigenvalues of the successive approximations M_l given by (3.10). First it is shown that the roots of the characteristic equation (1.19) coincide with the eigenvalues of matrix M from Theorem 3.3.4.

As a preparation, two lemmas are established about the existence and uniqueness of entire solutions of Equation (1.18) with certain exponential growth as $k \to \infty$.

Lemma 3.3.6. *Suppose* (3.1) *holds. Assume that* $\underline{x}_1[k]$ *and* $\underline{x}_2[k]$ *are entire solutions of* (1.18) *such that* $\underline{x}_1[0] = \underline{x}_1[0]$ *and*

$$\sup_{k\in\mathbb{Z}_-} \|\underline{x}_i[k]\|\nu_0^{-k} < \infty, \quad i = 1, 2. \tag{3.31}$$

Then $\underline{x}_1[k] = \underline{x}_1[k]$ *for all* $k \in \mathbb{Z}$.

Proof. Let

$$c = \sup_{k\in\mathbb{Z}_-} \|\underline{x}_1[k] - \underline{x}_2[k]\|\nu_0^{-k}. \tag{3.32}$$

By virtue of (3.31) , we have that $0 \le c < \infty$. From (1.18), we find for $k \in \mathbb{Z}_-$ and $i = 1, 2$,

$$\underline{x}_i[0] - \underline{x}_i[k] = \sum_{m=k}^{-1} \Delta \underline{x}_i[m] = \sum_{m=k}^{-1}\sum_{j=0}^{\infty} A[j]\underline{x}_i[m-j].$$

From this, (3.1) and (3.32), we obtain for $k \in \mathbb{Z}_-$,

$$\begin{aligned}\|\underline{x}_1[k] - \underline{x}_2[k]\| &\le \sum_{m=k}^{-1}\sum_{j=0}^{\infty} \|A[j]\|\|\underline{x}_1[m-j] - \underline{x}_2[m-j]\| \le \\ &\le c\sum_{m=k}^{-1}\sum_{j=0}^{\infty} \|A[j]\|\nu_0^{m-j} = c\sum_{m=k}^{-1}\nu_0^m\sum_{j=0}^{\infty}\|A[j]\|\nu_0^{-j} = \\ &= c\varkappa(1-\nu_0)\sum_{m=k}^{-1}\nu_0^m = c\varkappa\nu_0^k(1-\nu_0^{-k}) \le c\varkappa\nu_0^k,\end{aligned}$$

where

$$\varkappa = (1-\nu_0)^{-1}\sum_{j=0}^{\infty}\|A[j]\|\nu_0^{-j}.$$

Hence

$$\|\underline{x}_1[k] - \underline{x}_2[k]\|\nu_0^{-k} \le \varkappa c \quad \text{for all } k \in \mathbb{Z}_-$$

which yields $c \le \varkappa c$. By virtue of (3.1), we have that $\varkappa < 1$. Hence $c = 0$ and $\underline{x}_1[k] = \underline{x}_2[k]$ for all $k \in \mathbb{Z}_-$. From this, by the uniqueness of the solutions of the initial value problem (1.18) and (3.4), we obtain that $\underline{x}_1[k] = \underline{x}_2[k]$ identically for all $k \in \mathbb{Z}$. □

Lemma 3.3.7. *Suppose* (3.1) *holds. Then for every* $\underline{v} \in \mathbb{R}^n$, *Equation* (1.18) *has a unique entire solution* $\hat{\underline{x}} : \mathbb{Z} \to \mathbb{R}^n$ *such that* $\hat{\underline{x}}[0] = \underline{v}$ *and*

$$\sup_{k\in\mathbb{Z}_-} \|\hat{\underline{x}}[k]\|\nu_0^{-k} < \infty \tag{3.33}$$

given by

$$\hat{\underline{x}}[k] = M^k\underline{v} \quad \textit{for } k \in \mathbb{Z}, \tag{3.34}$$

where $M \in \mathbb{R}^{n\times n}$ *is the unique solution of the matrix equation* (3.8) *satisfying* (3.20).

Proof. The uniqueness of $\hat{\underline{x}}[k]$ follows from Lemma 3.3.6. In order to prove the existence, it is enough to show that the function $\hat{\underline{x}}[k]$ defined by (3.7) has the desired properties.

Let $\underline{\hat{x}}[k]$ defined by (3.7). It follows from (3.8) that $\underline{\hat{x}}[k]$ satisfies (1.18) for all $k \in \mathbb{Z}$. Furthermore, $\underline{\hat{x}}[0] = M^0\underline{v} = \underline{v}$. By virtue of (3.21), we have for $k \in \mathbb{Z}_-$,

$$\|M^k\| = \|(M^{-1})^{-k}\| \le \|M^{-1}\|^{-k} \le \nu_1^k \le \nu_0^k. \tag{3.35}$$

Hence

$$\|\underline{\hat{x}}[k]\| = \|M^k\underline{v}\| \le \|M^k\|\|\underline{v}\| \le \nu_0^k\|\underline{v}\| \quad \text{for } k \in \mathbb{Z}_-.$$

Thus, (3.33) also holds. □

Now the theorem for the approximation of the characteristic roots can be stated and proven.

Theorem 3.3.8. *Suppose* (3.1) *holds, and define*

$$\Lambda = \{\lambda \in \mathbb{C} \mid \det D(\lambda) = 0, |\lambda| \ge \nu_0\}, \tag{3.36}$$

where $D(\lambda)$ is the characteristic fucntion of (1.18) *defined by* (1.19). *Then*

$$\Lambda = \sigma(M), \tag{3.37}$$

where M is the unique solution of the matrix equation (3.8) *satisfying* (3.20).

Proof. Let $\lambda \in \Lambda$. Since $\det D(\lambda) = 0$, there exists a nonzero vector $\underline{v} \in \mathbb{R}^n$ such that $D(\lambda)\underline{v} = \underline{0}$. This implies that $\underline{\hat{x}} : \mathbb{Z} \to \mathbb{R}^n$ defined by $\underline{\hat{x}}[k] = \lambda^k\underline{v}$ for $k \in \mathbb{Z}$ is an entire solution of (1.18) with $\underline{\hat{x}}[0] = \underline{v}$. Furthermore, $|\lambda| \ge \nu_0$ implies for $t \in \mathbb{Z}_-$,

$$\|\underline{\hat{x}}[k]\| = |\lambda|^k\|\underline{v}\| \le \nu_0^k\|\underline{v}\|.$$

Thus, condition (3.33) of Lemma 3.3.7 is also satisfied. By the application of Lemma 3.3.7, we conclude that

$$\lambda^k\underline{v} = \underline{\hat{x}}[k] = M^k\underline{v} \quad \text{for all } k \in \mathbb{Z}.$$

In particular, $M\underline{v} = \lambda\underline{v}$ which implies that $\lambda \in \sigma(M)$. Thus $\Lambda \subset \sigma(M)$.

Next, let $\lambda \in \sigma(M)$. Then there exists a nonzero vector $\underline{v} \in \mathbb{R}^n$ such that $M\underline{v} = \lambda\underline{v}$. By Lemma 3.3.7, $\underline{\hat{x}}[k] = M^k\underline{v} = \lambda^k\underline{v}$ for $k \in \mathbb{Z}$, is an entire solution of (1.18) which implies that $D(\lambda)\underline{v} = \underline{0}$ and so $\det D(\lambda) = 0$. By virtue of (3.22), we have that $|\lambda| > \nu_0$ and hence $\lambda \in \Lambda$, i.e. $\sigma(M) \subset \Lambda$. □

According to Theorem 3.3.8, if (3.1) holds, then the roots of the characteristic equation (1.19) satisfying $|\lambda| \ge \nu_0$ coincide with the eigenvalues of M, the unique solution of (3.8) satisfying (3.20). Furthermore, by Theorem 3.3.5, M can be approximated by the sequence of matrices $\{M_l\}_{l=0}^\infty$ defined by (3.10). This, combined with Proposition 2.3.13, yields the following approximation theorem.

Theorem 3.3.9. *Suppose* (3.1) *holds so that the roots of the characteristic equation* (1.19) *satisfying $|\lambda| \ge \nu_0$ coincide with the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of M, the unique solution of the matrix equation* (3.8) *satisfying* (3.20). *If $\{M_l\}_{l=0}^\infty$ is the sequence defined by* (3.10), *then the eigenvalues $\lambda_1^{[l]}, \lambda_2^{[l]}, \ldots, \lambda_n^{[l]}$ of M_l can be renumbered such that the error estimate*

$$\max_{1 \le j \le n} |\lambda_j - \lambda_j^{[l]}| \le 8 \cdot 4^{-\frac{1}{n}} n^{\frac{1}{n}} (1 - \nu_1)\kappa^{\frac{l}{n}} \tag{3.38}$$

converges to 0 *exponentially fast as $l \to \infty$, where ν_1 and κ have the meaning from Theorem 3.3.5.*

Proof. Proposition 2.3.13 is applied with $P = M - I_n$ and $Q = M_l - I_n$. The eigenvalues of P and Q are $\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_n - 1$ and $\lambda_1^{[l]} - 1, \lambda_2^{[l]} - 1, \ldots, \lambda_n^{[l]} - 1$, respectively. By virtue of (3.20) and (3.27), we have that

$$\gamma = \max\{\|P\|, \|Q\|\} \leq 1 - \nu_1.$$

Proposition 2.3.13 implies that after renumbering the eigenvalues of M_l we have for all $j = 1, 2, \ldots, n$

$$|\lambda_j - \lambda_j^{[l]}| = |(\lambda_j - 1) - (\lambda_j^{[l]} - 1)| \leq 4 \cdot 2^{-\frac{1}{n}} n^{\frac{1}{n}} (2(1 - \nu_1))^{1-\frac{1}{n}} \|M - M_l\|^{\frac{1}{n}} \leq$$
$$\leq 4 \cdot 2^{-\frac{1}{n}} n^{\frac{1}{n}} (2(1 - \nu_1))^{1-\frac{1}{n}} ((1 - \nu_1)\kappa^l)^{\frac{1}{n}} = 8 \cdot 4^{-\frac{1}{n}} n^{\frac{1}{n}} (1 - \nu_1) \kappa^{\frac{l}{n}},$$

the last inequality being the consequence of (3.28). Thus, (3.38) holds. □

Theorem 3.3.9 can be illustrated by the following simple two-dimensional example.

Example 3.3.1. Consider the delay difference equation

$$\Delta \underline{x}[k] = A_1 \underline{x}[k-1], \quad \text{where } A_1 = \begin{pmatrix} 0 & \frac{6}{25} \\ \frac{6}{25} & 0 \end{pmatrix}. \tag{3.39}$$

Equation (3.39) is a special case of (1.18), when $n = 2$, $A[1] = A_1$ and $A[j] = O_2$ for $j \neq 1$. If $\|\cdot\| = \|\cdot\|_1$, then $\|A_1\| = 6/25 < 1/4$ and hence the assumption (3.1) is satisfied with $\nu_0 = 1/2$. The characteristic matrix from (1.19) has the form $D(z) = (z - 1)I_2 - A_1 z^{-1}$. It follows by easy calculations that the values of the quantities ν_1 and κ from Theorem 3.3.9 are $\nu_1 = 3/5$ and $\kappa = 2/3$. By the application of Theorems 3.3.8 and 3.3.9, it can be concluded that in the region $|z| \geq 1/2$ the characteristic equation $\det D(z) = 0$ has exactly two roots λ_1 and λ_2. Furthermore, the eigenvalues $\lambda_1^{[l]}$ and $\lambda_2^{[l]}$ of the successive approximations M_l given by

$$M_0 = I_2 \quad \text{and} \quad M_{l+1} = I_2 + A_1 M_l^{-1} \quad \text{for } l = 0, 1, \ldots$$

can be renumbered such that

$$\max_{j=1,2} |\lambda_j - \lambda_j^{[l]}| \leq \epsilon_l, \quad \text{where } \epsilon_l = \frac{8\sqrt{2}}{5} \left(\frac{2}{3}\right)^{k/2}. \tag{3.40}$$

Since

$$\det D(z) = z^{-2}(z + \frac{1}{5})(z - \frac{6}{5})(z - \frac{2}{5})(z - \frac{3}{5}),$$

the roots of the characteristic equation $\det D(z) = 0$ satisfying $|z| \geq 1/2$ are $\lambda_1 = 3/5$ and $\lambda_2 = 6/5$. The approximations $\lambda_1^{[l]}$ and $\lambda_2^{[l]}$ of λ_1 and λ_2 were computed in MATLAB (see Table). The numerical results are in full agreement with the error estimate (3.40).

The following example shows the properties of the scalar function (3.13) applied to Example 3.3.1.

Table 3.1 Approximation of the characteristic roots in Example 3.3.1

l	$\lambda_1^{[l]}$	$\lambda_2^{[l]}$	$\lvert\lambda_1 - \lambda_1^{[l]}\rvert$	$\lvert\lambda_2 - \lambda_2^{[l]}\rvert$	ϵ_l
1	0.76	1.24	0.16	0.04	1.8475
5	0.6192	1.2	$0.1925e-1$	$0.3001e-4$	0.8211
10	0.6023	1.2	$0.2339e-2$	$0.39e-8$	0.2980
15	0.6003	1.2	$0.3050e-3$	0	0.1081
20	0.6	1.2	$0.4011e-4$	0	$0.3924e-1$
25	0.6	1.2	$0.5280e-5$	0	$0.1424e-1$

Example 3.3.2. Consider the system from Example 3.6, which satisfies (3.1) with $\nu_0 = 0.8544$. The apropriate scalar function and its derivative are written as

$$f(\nu) = 1 - \nu - 0.025 - 0.025\nu^{-10}$$
$$f'(\nu) = -1 + 0.25\nu^{-11}.$$

The solution of $f'(\nu_2) = 0$ in the interval $\nu_2 > \nu_0$ is $\nu_2 = 0.8816$. The solutions of $f(\nu_1) = 0$ are 0.85543 and 0.9126 out of which only the latter satisfies $\nu_1 > \nu 2$.

Figure 3.1 shows the properties and the zero points of the scalar function and its derivative together with the maximum point of f. For better readability, the values of f are increased tenfold.

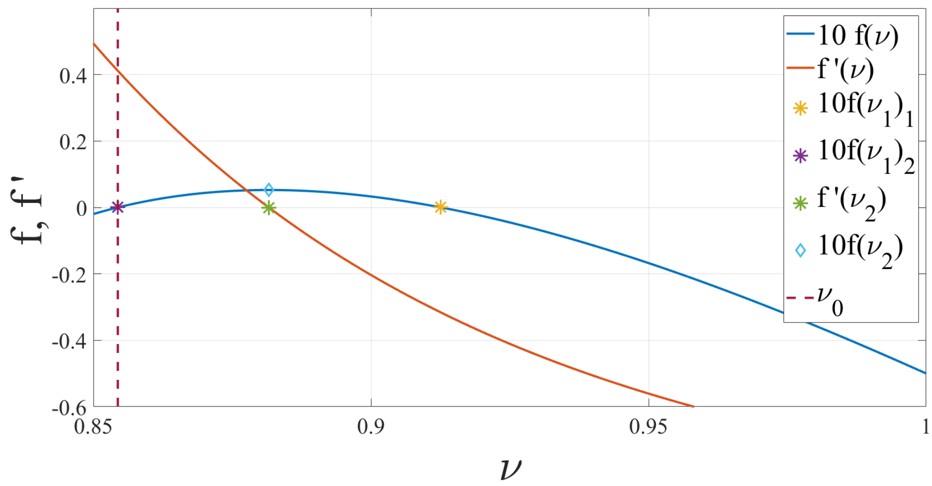

Figure 3.1 The function $f(\nu)$ and its derivative.

3.3.3 Asymptotic equivalence

In the following theorem, it is proven that under the smallness condition (3.1) the Volterra difference equation (1.18) is asymptotically equivalent to the ordinary difference equation (3.7).

Theorem 3.3.10. *Suppose* (3.1) *holds and let* M *be the unique solution of* (3.8) *satisfying* (3.20). *Then the following statements are valid:*
(i) *For every* $v \in \mathbb{R}^n$, *the function*

$$\hat{\underline{x}}_{\underline{v}}[k] = M^k \underline{v}, \quad \text{for } k \in \mathbb{Z}, \tag{3.41}$$

is a common entire solution for (1.18) *and* (3.7).
(ii) *For every solution* $\underline{x}[k]$ *of* (1.18) *corresponding to some initial function* $\phi \in \mathcal{B}_{\nu_0}$, *there exists a unique* $\underline{v} \in \mathbb{R}^n$ *such that*

$$\sup_{k \in \mathbb{Z}_+} \|\underline{x}[k] - \hat{\underline{x}}_{\underline{v}}[k]\| \nu_0^k < \infty \tag{3.42}$$

with $\hat{\underline{x}}_{\underline{v}}[k]$ *as in* (3.41). *In particular*

$$\lim_{k \to \infty} \|\underline{x}[k] - \hat{\underline{x}}_{\underline{v}}[k]\| = 0. \tag{3.43}$$

Proof. Conclusion *(i)* is a consequence of Lemma 3.3.7. Now we prove conclusion *(ii)*. First we show the existence of $\underline{v} \in \mathbb{R}^n$ such that (3.42) holds. Let $\underline{x}[k]$ be a solution of (1.18) corresponding to some initial function $\phi \in \mathcal{B}_{\nu_0}$. Denote by $\mathcal{Y}$ the set of those functions $\underline{y} : \mathbb{Z} \to \mathbb{R}^n$ which satisfy

$$\|\underline{y}\|_{\mathcal{Y}} = \sup_{k \in \mathbb{Z}} \|\underline{y}[k]\| \nu_0^{-k} < \infty. \tag{3.44}$$

$(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ is a Banach space. For $\underline{y}[k] \in \mathcal{Y}$, define the operator

$$(\mathcal{T}y)[k] = \begin{cases} -\sum_{m=k}^{\infty} \sum_{j=0}^{\infty} A[j]\underline{y}[m-j] & \text{if } k \geq 0, \\ \underline{x}[k] - M^k(\underline{x}[0] - (\mathcal{T}y)[0]) & \text{if } k < 0 \end{cases}. \tag{3.45}$$

We will show that $\mathcal{T}$ maps $\mathcal{Y}$ into itself. Let $\underline{y} \in \mathcal{Y}$. By virtue of (3.1) and (3.44), we have for $k \geq 0$,

$$\begin{aligned} \|(\mathcal{T}\underline{y})[k]\| &\leq \sum_{m=k}^{\infty} \sum_{j=0}^{\infty} \|A[j]\| \|\underline{y}[m-j]\| \leq \|\underline{y}\|_{\mathcal{Y}} \sum_{m=k}^{\infty} \sum_{j=0}^{\infty} \|A[j]\| \nu_0^{m-j} = \\ &= \|\underline{y}\|_{\mathcal{Y}} \sum_{m=k}^{\infty} \nu_0^m \sum_{j=0}^{\infty} \|A[j]\| \nu_0^{-j} \leq \|\underline{y}\|_{\mathcal{Y}} (1-\nu_0) \sum_{m=k}^{\infty} \nu_0^m = \|\underline{y}\|_{\mathcal{Y}} \nu_0^k. \end{aligned}$$

Hence

$$\sup_{k \geq 0} \|(\mathcal{T}y)[k]\| \nu_0^{-k} \leq \|\underline{y}\|_{\mathcal{Y}}. \tag{3.46}$$

In particular, $\|(\mathcal{T}y)[0]\| \leq \|\underline{y}\|_{\mathcal{Y}}$. From this, (3.3), (3.4) and (3.35), we find for $k < 0$,

$$\begin{aligned} \|(\mathcal{T}\underline{y})[k]\| = \|\phi[k] - M^k(\phi[0] - (\mathcal{T}\underline{y})[0])\| &\leq \|\phi[k]\| + \|M^k\|(\|\phi[0]\| + \|(\mathcal{T}\underline{y})[0]\|) \\ &\leq \nu_0^k \|\phi\|_{\mathcal{B}_{\nu_0}} + \nu_0^k(\|\phi\|_{\mathcal{B}_{\nu_0}} + \|\underline{y}\|_{\mathcal{Y}}) = (2\|\phi\|_{\mathcal{B}_{\nu_0}} + \|\underline{y}\|_{\mathcal{Y}})\nu_0^k. \end{aligned}$$

Hence

$$\sup_{k<0} \|(\mathcal{T}y)[k]\| \nu_0^{-k} \leq 2\|\phi\|_{\mathcal{B}_{\nu_0}} + \|\underline{y}\|_{\mathcal{Y}}. \tag{3.47}$$

From (3.46) and (3.47), we obtain that

$$\sup_{k\in\mathbb{Z}} \|(\mathcal{T}\underline{y})[k]\|\nu_0^{-k} \leq 2\|\phi\|_{\mathcal{B}_{\nu_0}} + \|\underline{y}\|_{\mathcal{Y}} < \infty. \tag{3.48}$$

Thus, $\mathcal{T}(\mathcal{Y}) \subset \mathcal{Y}$.

Next it is shown that $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$ is a contraction. Let $\underline{y}_1[k], \underline{y}_2[k] \in \mathcal{Y}$. For $k \geq 0$, we have

$$\begin{aligned}
\|(\mathcal{T}\underline{y}_1)[k] - (\mathcal{T}\underline{y}_2)[k]\| &\leq \sum_{m=k}^{\infty}\sum_{j=0}^{\infty} \|A[j]\|\|\underline{y}_1[m-j] - \underline{y}_2[m-j]\| \\
&\leq \|\underline{y}_1 - \underline{y}_2\|_{\mathcal{Y}} \sum_{m=k}^{\infty}\sum_{j=0}^{\infty} \|A[j]\|\nu_0^{m-j} = \|\underline{y}_1 - \underline{y}_2\|_{\mathcal{Y}} \sum_{m=k}^{\infty} \nu_0^m \sum_{j=0}^{\infty} \|A[j]\|\nu_0^{-j} \\
&= \|\underline{y}_1 - \underline{y}_2\|_{\mathcal{Y}} \frac{\nu_0^k}{1-\nu_0} \sum_{j=0}^{\infty} \|A[j]\|\nu_0^{-j} = \beta\|\underline{y}_1 - \underline{y}_2\|_{\mathcal{Y}}\nu_0^k,
\end{aligned}$$

where

$$\beta = \frac{1}{1-\nu_0}\sum_{j=0}^{\infty} \|A[j]\|\nu_0^{-j}.$$

Hence

$$\sup_{k\geq 0} \|(\mathcal{T}\underline{y}_1)[k] - (\mathcal{T}\underline{y}_2)[k]\|\nu_0^{-k} \leq \beta\|\underline{y}_1 - \underline{y}_2\|_{\mathcal{Y}}. \tag{3.49}$$

In particular, $\|(\mathcal{T}\underline{y}_1)[0] - (\mathcal{T}\underline{y}_2)[0]\| \leq \beta\|\underline{y}_1 - \underline{y}_2\|_{\mathcal{Y}}$. From this and (3.35), we find for $k < 0$,

$$\begin{aligned}
&\|(\mathcal{T}\underline{y}_1)[k] - (\mathcal{T}\underline{y}_2)[k]\| = \|M^k((\mathcal{T}\underline{y}_1)[0] - (\mathcal{T}\underline{y}_2)[0])\| \\
&\quad \leq \|M^k\|\|(\mathcal{T}\underline{y}_1)[0] - (\mathcal{T}\underline{y}_2)[0]\| \leq \nu_0^k\|(\mathcal{T}\underline{y}_1)[0] - (\mathcal{T}\underline{y}_2)[0]\| \leq \nu_0^k\beta\|\underline{y}_1 - \underline{y}_2\|_{\mathcal{Y}}.
\end{aligned}$$

Hence

$$\sup_{k<0} \|(\mathcal{T}\underline{y}_1)[k] - (\mathcal{T}\underline{y}_2)[k]\|\nu_0^{-k} \leq \beta\|\underline{y}_1 - \underline{y}_2\|_{\mathcal{Y}}. \tag{3.50}$$

From (3.49) and (3.50), we find that

$$\|\mathcal{T}\underline{y}_1 - \mathcal{T}\underline{y}_2\|_{\mathcal{Y}} \leq \beta\|\underline{y}_1 - \underline{y}_2\|_{\mathcal{Y}} \quad \text{whenever } \underline{y}_1, \underline{y}_2 \in \mathcal{Y}.$$

Since (3.1) implies that $\beta < 1$, $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$ is a contraction. By Banach's theorem (Appendix B.1.2), there exists $\underline{y} \in \mathcal{Y}$ such that $\mathcal{T}\underline{y} = \underline{y}$. By virtue of (3.45), the fixed point $\underline{y}$ of $\mathcal{T}$ satisfies

$$\Delta\underline{y}[k] = \sum_{j=0}^{\infty} A[j]\underline{y}[k-j] \quad \text{for } k \in \mathbb{Z}_+.$$

Hence $\underline{y}$ is a solution of (1.18) with initial condition

$$\underline{y}[k] = \underline{\phi}[k] - M^k\underline{v} \quad \text{for } k \in \mathbb{Z}_-,$$

where $\underline{v} = \underline{\phi}[0] - \underline{y}[0]$. Evidently, $\underline{z}[k] = \underline{x}[k] - M^k\underline{v}$, $k \in \mathbb{Z}$, is a solution of (1.18) with the same initial condition as $\underline{y}[k]$. Given the uniqueness of the solution of (1.18), we have that

$$\underline{y}[k] = \underline{z}[k] = \underline{z}[k] - M^k \underline{v} \quad \text{for all } k \in \mathbb{Z}.$$

This, together with (3.44), implies conclusion (3.42).

Finally, the uniqueness of $\underline{v} \in \mathbb{R}^n$ satisfying (3.42) is proven. Suppose for some $\underline{v}_1, \underline{v}_2 \in \mathbb{R}^n$,

$$\sup_{k \in \mathbb{Z}_+} \|\underline{x}[k] - \hat{\underline{x}}_{v_i}[k]\| \nu_0^{-k} = \sup_{k \in \mathbb{Z}_+} \|\underline{x}[k] - M^k \underline{v}_i\| \nu_0^{-k} < \infty \quad i = 1, 2. \tag{3.51}$$

By the triangle inequality, we have for $k \in \mathbb{N}^*$,

$$\|M^k(\underline{v}_1 - \underline{v}_1)\| = \|M^k \underline{v}_1 - M^k \underline{v}_1)\| \leq \|M^k \underline{v}_1 - \underline{x}[k]\| + \|\underline{x}[k] - M^k \underline{v}_1\|.$$

This, together with (3.51), implies that

$$c = \sup_{k \in \mathbb{Z}} \|M^k(\underline{v}_1 - \underline{v}_2)\| \nu_0^{-k} < \infty.$$

From this and (3.21), we find for $k \in \mathbb{Z}_+$,

$$\begin{aligned} \|\underline{v}_1 - \underline{v}_1\| = \|M^{-k} M^k (\underline{v}_1 - \underline{v}_1)\| &\leq \|M^{-k}\| \|M^k(\underline{v}_1 - \underline{v}_1)\| \\ &\leq \|M^{-1}\|^k c \nu_0^k \leq c \left(\frac{\nu_0}{\nu_1}\right)^k. \end{aligned}$$

Since $\nu_1 > \nu_0$, by letting $k \to \infty$ in the last inequality, we find that $\|\underline{v}_1 - \underline{v}_2\| = 0$ and hence $\underline{v}_1 = \underline{v}_2$. □

Recall some facts from the theory of ordinary difference equations [79]. If $\lambda \in \sigma(M)$ is an eigenvalue of $M \in \mathbb{R}^{n \times n}$, then its *algebraic multiplicity* $m_a(\lambda)$ is the multiplicity of $z = \lambda$ as a root of the characteristic polynomial $\det(zI_n - M)$. For every $\lambda \in \sigma(M)$, (3.7) has exactly $m_a(\lambda)$ linearly independent solutions of the form $\underline{p}[k]\lambda^k$, $k \in \mathbb{Z}$, where $\underline{p}[k]$ is a $\mathbb{R}^n$-valued polynomial in k of degree less than $m_a(\lambda)$. It is known that every solution of (3.7) is a sum of eigensolutions.

As shown in Theorem 3.3.4, if (3.1) holds and M is the solution of the matrix equation (3.8) satisfying (3.20), then, for every $\lambda \in \sigma(M)$, we have that $|\lambda| \geq \nu_1 > \nu_0$. This, combined with Theorem 3.3.10, yields the following result about the asymptotic representation of the solutions of the Volterra difference equation (1.18).

Corollary 3.3.11. *Suppose* (3.1) *holds. Let $\lambda_1, \lambda_2, \ldots, \lambda_l$ be the distinct eigenvalues of M, the unique solution of the matrix equation* (3.8) *satisfying* (3.20), *so that $|\lambda_j| \geq \nu_1 > \nu_0$, $1 \leq j \leq l$. If $\underline{x}[k]$ is a solution of* (1.18) *corresponding to some initial function $\phi \in \mathcal{B}_{\nu_0}$, then*

$$\underline{x}[k] = \sum_{j=1}^{l} \underline{p}_j[k] \lambda_j^k + O(\nu_0^k) \quad \text{as } k \to \infty, \tag{3.52}$$

where $\underline{p}_j[k]$ is a $\mathbb{R}^n$-valued polynomial in k of degree less than the order of $z = \lambda_j$ as a root of $\det(zI_n - M)$, $1 \leq j \leq l$.

3.4 Extension to non-homogeneous equations

This section shows that the proposed approximation method can be extended to input affine non-homogeneous difference systems with finite delays.

Theorem 3.4.1. *Consider the system*

$$\Delta \underline{x}[k] = \sum_{j=0}^{q} A[j]\underline{x}[k-j] + \underline{b}[k], \tag{3.53}$$

with $q \in \mathbb{N}$ and $\underline{b} : \mathbb{Z} \to \mathbb{R}^n$, which satisfies the smallness condition

$$\sum_{j=0}^{q} \|A[j]\|\nu_0^{-j} < 1 - \nu_0 \quad \text{for some} \quad \nu_0 \in (0,1) \tag{3.54}$$

and let

$$\underline{\hat{x}}[k+1] = M\underline{\hat{x}}[k] + \underline{\hat{b}}[k], \tag{3.55}$$

where M is the unique solution of (3.8) *with property* (3.20) *and $\underline{\hat{b}} : \mathbb{Z} \to \mathbb{R}^n$. If $\underline{\hat{b}}$ satisfies*

$$\underline{\hat{b}}[k] + \sum_{j=0}^{q} A[j] \sum_{l=k+q+1-j}^{k+q} M^{k+q-j-l}\underline{\hat{b}}[l-q-1] = \underline{b}[k], \quad \text{for all } k \in \mathbb{Z}, \tag{3.56}$$

then the following statements are valid:

1. *Every solution of* (3.55) *is a solution of* (3.53).
2. *For every solution $\underline{x}$ of* (3.53) *there exists a solution $\underline{\hat{x}}$ of* (3.55) *such that*

$$\underline{x}[k] - \underline{\hat{x}}[k] \to 0 \qquad \text{as } k \to \infty.$$

Proof. First we show that if (3.56) holds, then

$$\underline{\hat{x}}_P[k] = \sum_{l=0}^{k+q} M^{k+q-l}\underline{\hat{b}}[l-q-1] \tag{3.57}$$

is a common solution of (3.53) and (3.55). It is easy to verify that $\underline{\hat{x}}_P$ given by (3.57) satisfies (3.55). From (3.8) and (3.56), we find for $k \in \mathbb{Z}$,

$$\begin{aligned}
&\Delta\underline{\hat{x}}_P[k] - \sum_{j=0}^{q} A[j]\underline{\hat{x}}_P[k-j] \\
&\quad = \underline{\hat{b}}[k] + M\sum_{l=0}^{k+q} M^{k+q-l}\underline{\hat{b}}[l-q-1] - \sum_{l=0}^{k+q} M^{k+q-l}\underline{\hat{b}}[l-q-1] \\
&\qquad - \sum_{j=0}^{q} A[j]M^{-j} \sum_{l=0}^{k-j+q} M^{k+q-l}\underline{\hat{b}}[l-q-1] \\
&\quad = \underline{\hat{b}}[k] + \underbrace{\left(M - I - \sum_{j=0}^{q} A[j]M^{-j}\right)}_{=O_n} \sum_{l=0}^{k+q} M^{k+q-l}\underline{\hat{b}}[l-q-1] \\
&\qquad\qquad + \sum_{j=0}^{q} A[j] \sum_{l=k-j+q+1}^{k+q} M^{k-j+q-l}\underline{\hat{b}}[l-q-1].
\end{aligned}$$

Thus, $\underline{\hat{x}}_P$ given by (3.56) is a solution of (3.53) if and only if (3.56) holds. Since every solution $\underline{x}$ of (3.55) has the form $\underline{\hat{x}} = \underline{\hat{x}}_H + \underline{\hat{x}}_P$, where $\underline{\hat{x}}_H$ is a solution of (3.7) and hence, by Theorem 3.3.10, of (1.18), it follows that $\underline{x}$ is a solution of (3.53).

Now suppose that $\underline{x}$ is an arbitrary solution of (3.53). Then $\underline{x}_H = \underline{x} - \underline{\hat{x}}_P$ is a solution of (1.18). By Theorem 3.3.10, there exists a solution $\underline{\hat{x}}_H$ of (3.7) such that $\lim_{k\to\infty} \|\underline{x}_H[k] - \underline{\hat{x}}_H[k]\| = 0$. Evidently, $\underline{\hat{x}} = \underline{\hat{x}}_H + \underline{\hat{x}}_P$ is a solution of (3.7) such that $\lim_{k\to\infty} \|\underline{x}[k] - \underline{\hat{x}}[k]\| = \lim_{k\to\infty} \|\underline{x}_H[k] - \underline{\hat{x}}_H[k]\| = 0$.

□

If the non-homogeneous term in (3.53) is constant i.e. $\underline{b}[k] \equiv \underline{b}$, then Equation (3.56) admits a constant solution given by

$$\underline{\hat{b}} = \left(I - \sum_{j=0}^{q} A[j] \sum_{l=q-j+1}^{q} M^{q-j-l} \right)^{-1} \underline{b}. \tag{3.58}$$

3.5 Approximation of MAS with delay

In this section, it will be shown that the modelling of MAS with communication delays and apply the analysis method shown in the previous sections.

3.5.1 Model of MAS

Consider a MAS consisting of agents with unique identifiers $\mathcal{I} = \{1, 2, \ldots, n\}$, $n \in \mathbb{N}^*$. The interconnection among the agents are described by an undirected graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$. Each vertex of the set $\mathcal{V} = \mathcal{I}$ represents an agent. The elements of the edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ show the interconnections among the agents $(i, j) \in \mathcal{E}$ if the vertices i, and j are connected. It is assumed that $(i, i) \notin \mathcal{E}$. The neighbour set of the i^{th} vertex is $\mathcal{N}_i = \{j \in \mathcal{V} \mid (i, j) \in \mathcal{E}\}$.

The dynamics of an agent is described by the linear difference equations

$$\Delta x_i[k] = u_i[k], \qquad 1 \le i \le n, \tag{3.59}$$

where $x_i[k] : \mathbb{Z} \to \mathbb{R}$ denotes the state of the i^{th} agent, and the input $u_i[k]$ will be specified later.

The MAS reaches consensus if $\lim_{k\to\infty}(x_i[k] - x_j[k]) = 0$ for all $i, j = 1, \ldots, n$.

According to the consensus protocol, the input $u_i[k] : \mathbb{Z} \to \mathbb{R}$ of the agent is formulated as

$$u_i[k] = \frac{\epsilon}{N} \sum_{j \in \mathcal{N}_i} \bigg(x_j[k-q] - x_i[k] \bigg), \qquad 1 \le i \le n, \tag{3.60}$$

where $0 < \epsilon < 1$, $N = \max_i |\mathcal{N}_i|$, and $q \in \mathbb{N}^*$ is the communication delay.

The communication graph must be connected to achieve consensus. For more details about the consensus theory of discrete-time MAS with delay, see [94], [92].

The overall model of the MAS with communication delay is given by

$$\Delta \underline{x}[k] = -D\underline{x}[k] + A\underline{x}[k-q], \tag{3.61}$$

with state vector $\underline{x}[k] = \begin{pmatrix} x_1[k] & \cdots & x_n[k] \end{pmatrix}^T$ and initial data $\underline{x}[h] = \begin{pmatrix} \theta_1 & \cdots & \theta_n \end{pmatrix}^T$, for $h = -q, \ldots, 0$. The adjacency matrix $A \in \mathbb{R}^{n\times n}$ is defined by $a_{ij} = \epsilon/N$ if $(i, j) \in \mathcal{E}$ (otherwise, $a_{ij} = 0$). The degree matrix of the graph is given by $D = (\epsilon/N)\mathrm{diag}(|\mathcal{N}_1| \ldots |\mathcal{N}_n|)$.

In many applications, such as robot platooning, it is essential to bias the steady-state consensus value. For example, in robotic systems, the bias term can represent an external command for the leader robot.

The input with bias term can be formulated as

$$u_i[k] = \frac{\epsilon}{N} \sum_{j \in \mathcal{N}_i} \Big(x_j[k-q] - x_i[k] \Big) + b_i[k], \quad 1 \leq i \leq n, \tag{3.62}$$

where $\underline{b}_i : \mathbb{R}_+ \to \mathbb{R}$.

In this case the global model has the form

$$\Delta \underline{x}[k] = -D\underline{x}[k] + A\underline{x}[k-q] + \underline{b}[k], \tag{3.63}$$

where $\underline{b} : \mathbb{R}_+ \to \mathbb{R}^n$ is defined by the entries $\underline{b}[k] = \big(b_1[k] \cdots b_n[k]\big)^T$.

3.5.2 Application to MAS with communication delay

In Section 3.3 we have developed an approximation method for homogeneous Volterra difference systems with infinite delays. Equation (1.18) includes as a special case the delay difference equation

$$\Delta \underline{x}[k] = -D\underline{x}[k] + A\underline{x}[k-q], \tag{3.64}$$

where $q \in \mathbb{N}^*$, $D \in \mathbb{R}^{n \times n}$, $A \in \mathbb{R}^{n \times n}$, with $A_q \neq O_n$. In this case the condition (3.1) has the form

$$\|D\| + \|A\|\nu_0^{-q} < 1 - \nu_0 \quad \text{for some } \nu_0 \in (0,1). \tag{3.65}$$

It is a simple exercise in calculus to show that condition (3.65) holds if and only if

$$\|D\| < 1 \quad \text{and } \|A\| < \frac{q^q}{(q+1)^{(q+1)}}(1 - \|D\|)^{q+1}. \tag{3.66}$$

Thus, the results from Section 3.3 apply to (3.64) under the explicit smallness condition (3.66).

3.5.3 Approximation of MAS with communication delay

Problem statement:

Consider the MAS with the dynamics (3.63). Determine an ordinary difference equation of the form (3.55), such that equations (3.55) and (3.63) are asymptotically equivalent and the dominant eigenvalues of (3.64) are the eigenvalues of (3.55).

Approximation without non-homogeneous term

Corollary 3.5.1. *If*

$$\frac{q^q}{(q+1)^{q+1}}(1-\epsilon)^{q+1} - \epsilon > 0, \tag{3.67}$$

then system (3.64) *is asymptotically equivalent to* (3.7) *in the sense of Theorem 3.3.10.*

Proof. Let $\|\cdot\| = \|\cdot\|_1$. Then $\|D\| = \|A\| = \epsilon$ and (3.66) reduces to (3.67). The result follows from Theorem 3.3.10. □

Figure 3.2 shows the feasible ϵ and discrete delay value pairs for which the approximation remains valid. Note that the communication topology of MAS does not influence the smallness condition (3.67).

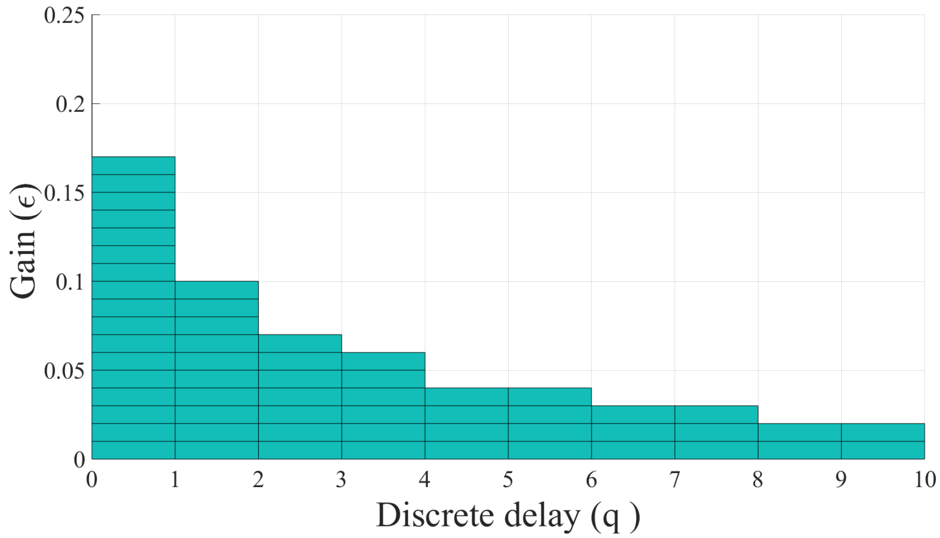

Figure 3.2 Feasible values of discrete delay and gain pairs based on the smallness condition (3.66).

State matrix computation
The matrix equation (3.8) in the case of (3.64) becomes

$$M = I_n - D + AM^{-q}, \tag{3.68}$$

which can be computed as a limit the iterations

$$M_0 = I_n \quad M_{l+1} = I_n - D + AM_l^{-q}, \quad \text{for } l = 0, 1, \ldots. \tag{3.69}$$

Dominant eigenvalues
The characteristic equation of (3.64) is given by

$$\det(\lambda I_n - I_n + D - A\lambda^{-q}) = 0. \tag{3.70}$$

If $I_n - D + A$ is row stochastic, then $\lambda = 1$ is an eigenvalue of (3.64).

According to Theorem 3.3.8, the dominant eigenvalues of (3.64) satisfy $|\lambda| > \nu_0$. Since $\nu_0 < 1$, $\lambda = 1$ is a dominant eigenvalue, and hence it is also an eigenvalue of M.

It is easy to see that in this case the system matrix obtained by (3.68) is also a row stochastic matrix.

Approximation with non-homogeneous term
If the consensus protocol contains a bias, the approximation can be extended using the results developed in Section 3.4. By using (3.58) and (3.68) with the notation $A[0] = -D$, $A[q] = A$ and $A[j] = O_n$ for $j = 1, 2, \ldots, q-1$ the approximated bias term yields

$$\hat{\underline{b}} = \left(I_n + A \sum_{l=1}^{q} M^{-l} \right)^{-1} \underline{b}. \tag{3.71}$$

3.6 Case studies

Homogeneous MAS:

Figure 3.3 presents an example of MAS consisting of 5 agents with connection topology given by an undirected, complete graph. The discrete-time communication

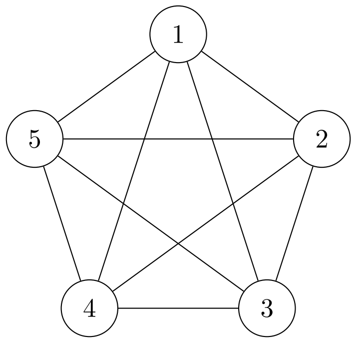

Figure 3.3 The communication topology of the MAS.

delay is $q = 10$ samples and $\epsilon = 0.025$, and each agent implements the consensus protocol (3.60). In the global model of the MAS the degree matrix is $D = \epsilon I$, the adjacency matrix is $A = (a_{ij})$, with $a_{ij} = \epsilon/4$ if $i \neq j$, 0 otherwise.

The smallness condition is satisfied with $\nu_0 = 0.855$. The system matrix M_l was computed using the iteration (3.69), with an iteration accuracy $\|M - M_l\| \leq 6.5036e - 06$.

To evaluate the numerical results, the eigenvalues of the delayed MAS are also calculated by applying the Matlab Symbolic Toolbox to the relation (3.70).
Figure 3.4 shows that the eigenvalues of M_l, 1 and 0.9662 with a multiplicity of 4 match the dominant eigenvalues of the delayed MAS.

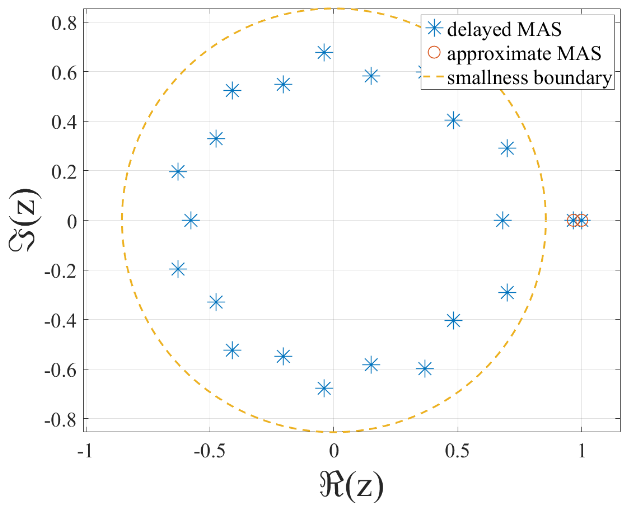

Figure 3.4 The eigenvalues of the delayed MAS and the approximate MAS.

Furthermore, Figure 3.5 shows that the trajectories of the approximated MAS converge asymptotically to the trajectories of the MAS with delay. The relative trajectory error between the delayed and approximate MAS, calculated in each discrete step as $e[k] = \frac{\|x[k]-\hat{x}[k]\|}{\|x[k]\|} \cdot 100$, is always below 0.6%, as shown in Figure 3.6.

Non-homogeneous MAS:

Figure 3.7 presents an example of MAS consisting of 6 agents in an undirected graph.

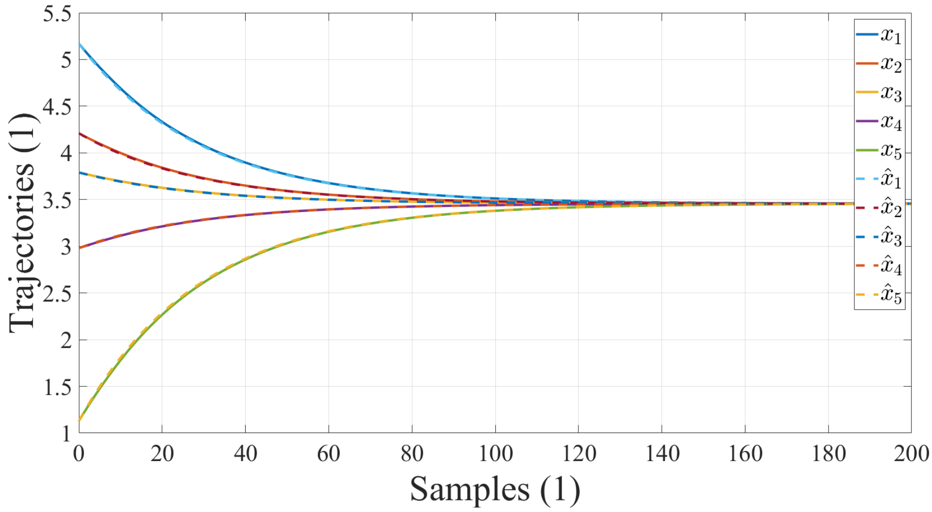

Figure 3.5 The trajectories of the delayed MAS and the approximate MAS.

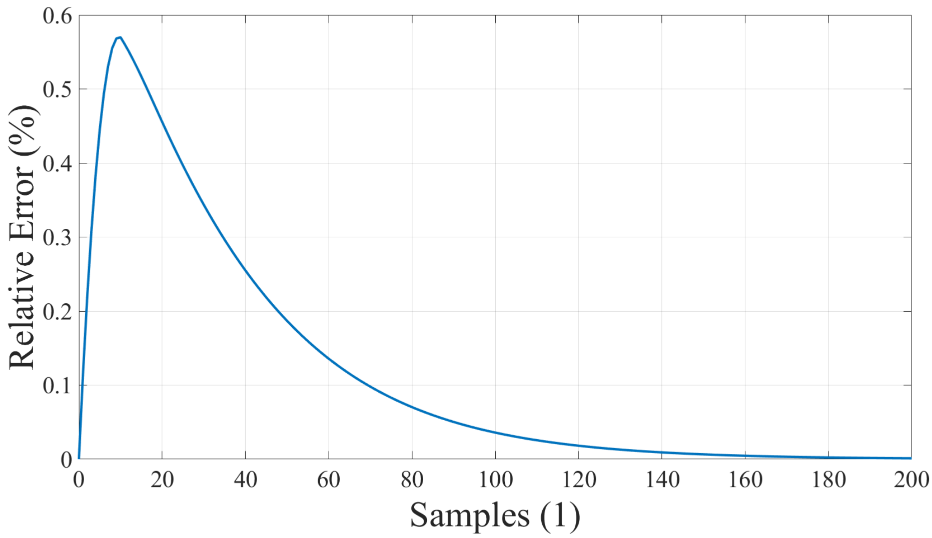

Figure 3.6 The relative trajectory error between the delayed MAS and the approximate MAS.

The discrete-time communication delay is $q = 5$, the gain is $\epsilon = 0.048$, and the offset is $\underline{b} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T$, i.e the agent in node 1 has a unit bias term in its input. The global system dynamics is described by the relation (3.63), where the degree matrix is $D = \epsilon I$, the adjacency matrix is $A = (a_{ij})$, where $a_{12} = a_{13} = a_{43} = a_{46} = \epsilon/2$, $a_{21} = a_{53} = a_{64} = \epsilon$, $a_{31} = a_{34} = a_{35} = \epsilon/3$ and $a_{ij} = 0$ otherwise .

The smallness condition is satisfied with $\nu_0 = 0.751$. The system matrix M_l was computed using the iteration (3.69), with an iteration accuracy $\|M - M_l\| \leq 2.9826e-06$.

Figure 3.8 shows that the eigenvalues of M_l, 0.8305, 0.8918, 0.9227, 0.9743, 0.9880 and 1, match the dominant eigenvalues of the delayed MAS.

The offset of the approximate system can be calculated using the relation (3.71), which yields $\hat{\underline{b}} = \begin{pmatrix} 1.159 & -0.387 & -0.172 & 0.074 & 0.105 & -0.055 \end{pmatrix}^T$. Figure 3.9 shows

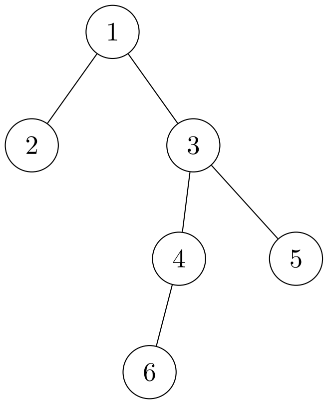

Figure 3.7 The communication topology of the MAS with 6 agents and a single leader.

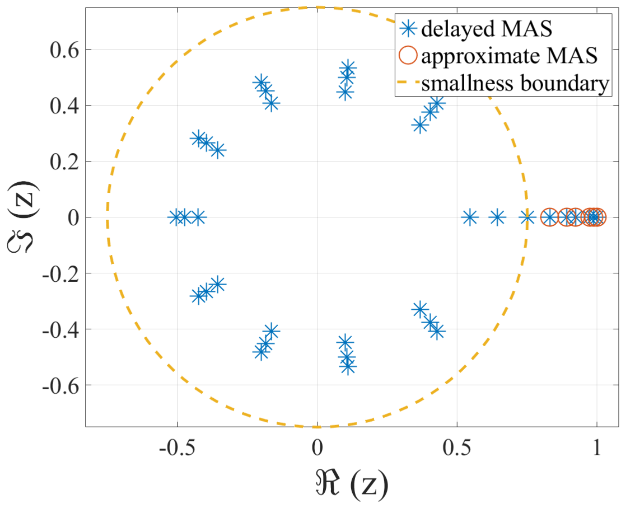

Figure 3.8 The eigenvalues of the delayed MAS and the approximate MAS.

that the trajectories of the approximated MAS converge asymptotically to the trajectories of the MAS with delay. The relative error between the delayed and approximate MAS is always below 8.4%, as shown in Figure 3.10.

3.7 Summary

The communication delay influences both the transient and steady-state behaviour of MASs. In this chapter, it was shown that for discrete-time MASs with small gains, not just the delay-induced steady-state bias can be computed, but such an approximate system can be determined whose trajectories asymptotically converge to the trajectories of the original delay system.

The proposed procedure is based on a general approximation method initially developed for infinite-dimensional Volterra difference equations. The approximate system has the same state dimension (number of equations) as the original delay system. It preserves the dominant eigenvalues of the delay system, and its state matrix can be computed using an iterative numerical method.

The approximation method was extended to systems with non-homogeneous terms. It was also shown that the non-homogeneous approximation can be applied as an approximation of MASs with agents that inputs contain bias terms.

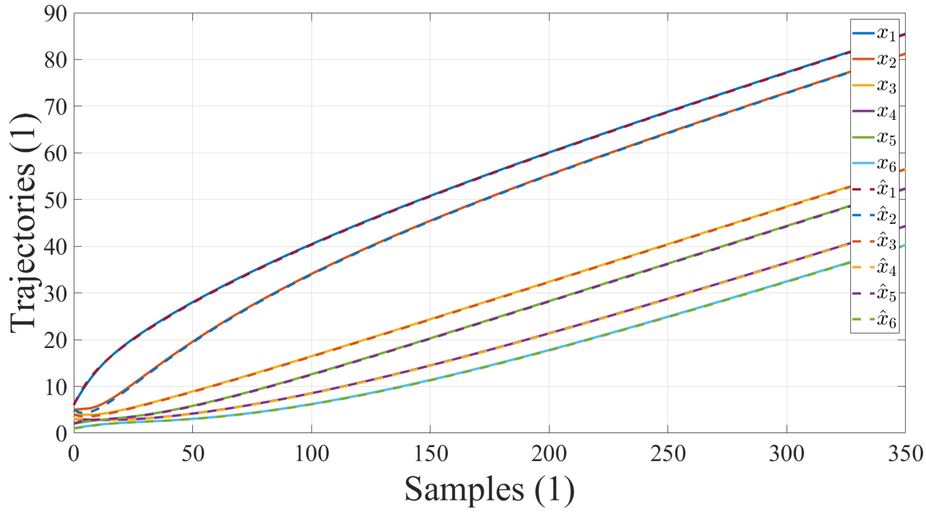

Figure 3.9 The trajectories of the delayed MAS and the approximate MAS.

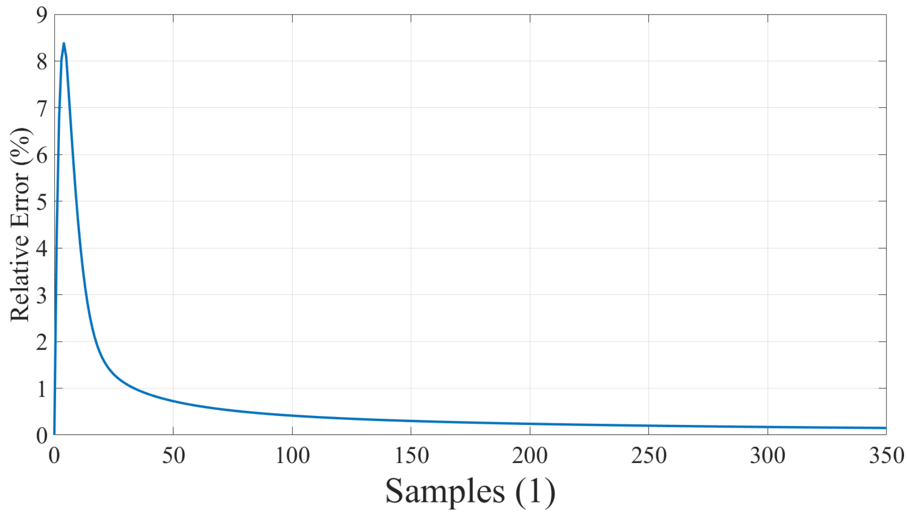

Figure 3.10 The relative trajectory error between the delayed MAS and the approximate MAS.

The numerical evaluation of the developed method shows that the delay-free system can well approximate the trajectories and dominant eigenvalues of the delay system.

Chapter 4

Approximation of linear systems with distributed delays

4.1 Abstract

This chapter shows that a linear delay-free system can approximate a linear system with distributed state delay with the same state dimension as the original system if a so-called smallness condition holds. The eigenvalues of the approximate system correspond to the dominant eigenvalues of the original system with distributed delay. A numerically stable, iterative algorithm is provided to compute the state matrix of the approximate system. Furthermore, it is shown that, based on the proposed approximation, the stabilisation, pole placement, and setpoint tracking control problems of the addressed class of distributed delay systems can be performed using methods developed for delay-free systems. Simulation results are provided to show the applicability of the proposed approximation and control design method.

This chapter is based on [97*].

4.2 Literature survey

Many physical phenomena can efficiently be modelled using differential equations that contain distributed delay, i.e. Delay Integro-Differential Equation (DIDE). Such models can be used to describe population growth [98], chemical reaction networks in the presence of the spacially distributed convection [99], epidemiology processes [100, 101], cell maturation under multiple growth-factor effects [102, 103], traffic flow dynamics [104], human driver behavior [105] or mechanical engineering systems [106].

It is known that DIDEs are special cases of infinite-dimensional systems. In the case of linear DIDEs, the eigenvalues form an infinite spectrum, and there exist infinitely many eigensolutions [107]. There are are several methods for the numerical approximation of DIDEs, e.g. Taylor polynomial based continuous collocation method [108], spline approximation [109], rational approximation [110], Chebyshev polynomial based Galerkin's method [111] or Legendre polynomial based Galerkin's method with tau incorporation [112].

The rightmost eigenvalues of DIDEs determine the dominant dynamic behaviour of these systems. If these eigenvalues are properly manipulated using feedback control, desired dynamic behaviour can be achieved. However, during the control design for systems with distributed delay, several problems can arise, such as a computationally expensive design procedure or lack of numerical convergence [113]. The stabilisation problem of systems with distributed delay was traced back to an LMI (Linear Matrix Inequality) problem by Feng, and Nguang [114]. Zhong [115] proposed frequency domain approaches for efficient approximation and implementation of controlled distributed delay systems. Vidyasagar and Anderson [116] gave sufficient and necessary conditions for a system containing countable delay distributions terms to be approximated by a lumped system.

In this chapter, a constructive approximation method is proposed for a class of distributed delay systems. The approximation model has the same number of equations as the original system, and its eigenvalues are the same as the dominant eigenvalues of the original delay system. The approximation can also be applied to systems having additive non-homogeneous terms. It is also shown that the stabilisation and setpoint tracking control of the addressed class of distributed delay system can be solved using design methods developed for delay-free systems.

The sophisticated spline-based approximation method presented in [117] uses high dimensional systems of ODE for the approximation. It should be noted that the Galerkin's [111] and spline-based approximation methods have the advantage that there is no restriction for the size of the delays. However, the dimensions of the state space of the approximation model in these cases are larger than the dimension of the state space of the original delay system. The approximation proposed in this study preserves the dimension of the state space.

4.3 Approximation of the homogeneous equation

Consider the homogeneous system (1.20). Throughout the chapter it is assumed that the relation

$$\|A_0\| + \int_{-\tau}^{0} \|A_\tau(\eta)\| \mathrm{e}^{-\eta\nu} \mathrm{d}\eta < \nu, \tag{4.1}$$

holds for some $\nu > 0$, which may be viewed as a *smallness condition* on the maximum delay τ.

It will be shown that if (4.1) holds, then (1.20) is asymptotically equivalent to the ODE

$$\dot{\underline{x}}(t) = M\underline{x}(t), \tag{4.2}$$

where $M \in \mathbb{R}^{n\times n}$ is the unique solution of the matrix equation

$$M = A_0 + \int_{-\tau}^{0} A_\tau(\eta) \mathrm{e}^{M\eta} \mathrm{d}\eta \tag{4.3}$$

such that

$$\|M\| < \nu_0, \tag{4.4}$$

where $\nu_0 > 0$ is the unique solution of the scalar equation

$$\int_{-\tau}^{0} (-\eta) \|A_\tau(\eta)\| \mathrm{e}^{-\nu_0\eta} \mathrm{d}\eta = 1. \tag{4.5}$$

Furthermore, the system matrix M in (4.2) can be written as a limit of successive approximations

$$M = \lim_{k\to\infty} M_k, \tag{4.6}$$

where

$$M_0 = O_n \quad \text{and} \quad M_{k+1} = A_0 + \int_{-\tau}^{0} A_\tau(\eta) \mathrm{e}^{M_k\eta} \mathrm{d}\eta, \qquad \text{for } k = 0, 1, \ldots. \tag{4.7}$$

The convergence in (4.6) is exponential and an estimate is given for the approximation error $\|M - M_k\|$. It will be shown that those characteristic roots of (1.20) which lie in the half plane $\Re(\lambda) > -\nu_0$, coincide with the eigenvalues of matrix M. As a consequence, the above dominant characteristic roots of (1.20) can be approximated by the eigenvalues of M_k. An explicit estimate is given for the approximation error, which shows that the convergence of the eigenvalues of M_k to the dominant characteristic roots of (1.20) is exponentially fast.

4.3.1 Solution of the associated matrix equation

First some properties of the scalar equation

$$\nu - \|A_0\| - \int_{-\tau}^{0} \|A_\tau(\eta)\| e^{-\nu\eta} d\eta = 0 \tag{4.8}$$

are needed.

Lemma 4.3.1. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be given by*

$$f(\nu) = \nu - \|A_0\| - \int_{-\tau}^{0} \|A_\tau(\eta)\| e^{-\nu\eta} d\eta. \tag{4.9}$$

and suppose that (4.1) *holds. Then there exists a unique $\nu_0 > 0$ for which $f'(\nu_0) = 0$ and there exists a unique $\nu_1 \in (0, \nu_0)$ such that $f(\nu_1) = 0$. Moreover, $f(\nu) > 0$ for all $\nu \in (\nu_1, \nu_0]$, and $f'(\nu) > 0$ for all $\nu \in [\nu_1, \nu_0)$, or equivalently,*

$$\|A_0\| + \int_{-\tau}^{0} \|A_\tau(\eta)\| e^{-\nu\eta} d\eta < \nu \qquad \text{for } \nu \in (\nu_1, \nu_0] \tag{4.10}$$

and

$$\int_{-\tau}^{0} (-\eta)\|A_\tau(\eta)\| e^{-\nu\eta} d\eta < 1 \qquad \text{for } \nu \in [\nu_1, \nu_0). \tag{4.11}$$

Proof. From the definition of $f(\nu)$, we find that

$$f'(\nu) = 1 - \int_{-\tau}^{0} (-\eta)\|A_\tau(\eta)\| e^{-\nu\eta} d\eta,$$

$$f''(\nu) = - \int_{-\tau}^{0} \eta^2 \|A_\tau(\eta)\| e^{-\nu\eta} d\eta.$$

It can be seen that $f''(\nu) < 0$ on $\mathbb{R}_+^*$, so $f'(\nu)$ is strictly decreasing and $\lim_{\nu\to\infty} f'(\nu) = -\infty$.

If $f'(0) \leq 0$ would hold, then $f'(\nu) \leq 0 \quad \forall \nu > 0$, so $f(\nu)$ is decreasing and

$$f(0) = -\|A_0\| - \int_{-\tau}^{0} \|A_1(\eta)\| d\eta \geq f(\nu) \quad \text{for } \nu > 0.$$

Since $f(0) < 0$ we have $f(\nu) < 0$ for all $\nu > 0$, which together with (4.1) leads to a contradiction. Accordingly $f'(0) > 0$.

Since $f'(\nu)$ is strictly decreasing, $f'(0) > 0$, $f'(\infty) = -\infty$, it follows from the properties of the continuous functions that there exists a unique $\nu_0 > 0$ such that $f'(\nu_0) = 0$.

Since $f'(\nu) > 0$ on $[0, \nu_0)$, and $f'(\nu) < 0$ on (ν_0, ∞) it can be stated that $f(\nu)$ is strictly increasing on $[0, \nu_0]$, and $f(\nu)$ is strictly decreasing on $[\nu_0, \infty)$. Hence $f(\nu)$ has a global maximum at ν_0.

We have that $f(0) < 0$ and, by (4.1),

$$f(\nu_0) = \nu_0 - \|A_0\| - \int_{-\tau}^{0} \|A_\tau(\eta)\| \mathrm{e}^{-\nu_0 \eta} \mathrm{d}\eta > 0.$$

Since $f(\nu)$ is strictly increasing on $[0, \nu_0]$, $f(0) < 0$, and $f(\nu_0) > 0$, there exists a unique $\nu_1 \in (0, \nu_0)$ such that $f(\nu_1) = 0$. Furthermore, $f(\nu) > 0$ on $(\nu_1, \nu_0]$, and $f'(\nu) > 0$ on $[\nu_1, \nu_0)$. □

Now the theorem regarding the existence and uniqueness of the solution of the exponential matrix equation (4.3) can be stated.

Theorem 4.3.2. *Suppose that* (4.1) *holds. Then the equation* (4.3) *has a unique solution* $M \in \mathbb{R}^{n\times n}$ *such that* (4.4) *holds, with* ν_0 *as in Lemma 4.3.1.*

Proof. Let $\nu \in [\nu_1, \nu_0)$ be fixed, where ν_1 is defined in Lemma 4.3.1, and define

$$\mathcal{S} = \mathcal{S}_\nu = \{M \in \mathrm{R}^{n\times n} \mid \|M\| \leq \nu\}.$$

Define $F : \mathcal{S} \to \mathbb{R}^{n\times n}$ by

$$F(M) = A_0 + \int_{-\tau}^{0} A_\tau(\eta) \mathrm{e}^{M\eta} \mathrm{d}\eta.$$

Clearly $\mathcal{S} \neq \emptyset$.

For all $M \in \mathcal{S}$ we have

$$\begin{aligned} \|F(M)\| \leq \|A_0\| + \int_{-\tau}^{0} \|A_\tau(\eta)\| \|\mathrm{e}^{M\eta}\| \mathrm{d}\eta &\leq \|A_0\| + \int_{-\tau}^{0} \|A_\tau(\eta)\| \mathrm{e}^{\|M\||\eta|} \mathrm{d}\eta \\ &\leq \|A_0\| + \int_{-\tau}^{0} \|A_\tau(\eta)\| \mathrm{e}^{-\nu\eta} \mathrm{d}\eta. \end{aligned}$$

By virtue of (4.10), we have that $\|F(M)\| \leq \nu$, i.e. F is a self-mapping of $\mathcal{S}$.

Let $M_1, M_2 \in \mathcal{S}$. We can write the difference

$$F(M_1) - F(M_2) = \int_{-\tau}^{0} A_\tau(\eta) (\mathrm{e}^{M_1\eta} - \mathrm{e}^{M_2\eta}) \mathrm{d}\eta.$$

From this, we get

$$\|F(M_1) - F(M_2)\| \leq \int_{-\tau}^{0} \|A_\tau(\eta)\| \|\mathrm{e}^{M_1\eta} - \mathrm{e}^{M_2\eta}\| \mathrm{d}\eta.$$

It is known that

$$\|\mathrm{e}^{M_1\eta} - \mathrm{e}^{M_2\eta}\| \leq \mathrm{e}^{\gamma} \|M_1 - M_2\| |\eta|,$$

where $\gamma = \max\{\|M_1\|, \|M_2\|\}|\eta| \leq \nu|\eta|$. So the norm of the difference can be estimated by

$$\|F(M_1) - F(M_2)\| \leq \underbrace{\int_{-\tau}^{0} (-\eta) \|A_\tau(\eta)\| \mathrm{e}^{-\nu\eta} \mathrm{d}\eta}_{\kappa} \|M_1 - M_2\|.$$

By virtue of (4.11), we have that $\kappa < 1$, which means that $F : \mathcal{S} \to \mathcal{S}$ is a contraction. Hence there exists a unique $M \in \mathcal{S}$ such that $M = F(M)$ according to Banach's fixed point theorem (Appendix B.1.2). Since $\nu \in [\nu_1, \nu_0)$ was arbitrary, this proves the theorem. □

In the following theorem, it is shown that the unique solution of the matrix equation (4.3) with property (4.4) can be written as a limit of successive approximations M_k defined by (4.7) and an estimate is given for the approximation error.

Theorem 4.3.3. *Suppose* (4.1) *holds. Let* $M \in \mathbb{R}^{n\times n}$ *be the unique solution of* (4.3) *satisfying* (4.4). *If* $\{M_k\}_{k=0}^{\infty}$ *is the sequence of matrices defined by* (4.7)*, then*

$$\|M_k\| \leq \nu_1 \quad \text{for all } k \in \mathbb{N},$$

and the approximation error in the k^{th} *step is given by*

$$\|M - M_k\| < \kappa^k \nu_1 \qquad \text{for } k \in \mathbb{N}, \tag{4.12}$$

where

$$\kappa = \int_{-\tau}^{0} (-\eta)\|A_1(\eta)\|e^{-\nu_1\eta}d\eta < 1, \tag{4.13}$$

with ν_1 *as in Lemma 4.3.1.*

Proof. As a consequence of Theorem 4.3.2 the iteration $M_{k+1} = F(M_k)$ with $M_k \in \mathcal{S}_{\nu_1}$ converges to the fixed point $M = F(M)$ for $k \to \infty$. Moreover, $\|M\| \leq \nu_1$, and $\|M_k\| \leq \nu_1$, for all $k \in \mathbb{N}$.

The norm $\|M - M_{k+1}\|$ can be estimated by applying (4.3), (4.7), and Lemma 2.3.2 as follows:

$$\|M - M_{k+1}\| \leq \int_{-\tau}^{0} \|A_1(\eta)\|\|e^{\eta M} - e^{\eta M_k}\|d\eta \leq \int_{-\tau}^{0} (-\eta)\|A_1(\eta)\|e^{-\nu_1\eta}d\eta\|M - M_k\|.$$

Tis, together with (4.13), yields

$$\|M - M_{k+1}\| \leq \kappa\|M - M_k\|, \quad \text{for } k \in \mathbb{N}$$

from which it follows by induction on k that

$$\|M - M_k\| \leq \kappa^k\|M - M_0\| \leq \kappa^k \nu_1$$

for $k \in \mathbb{N}$. □

4.3.2 Dominant eigenvalues and eigensolutions

Next, the asymptotic, delay-free approximation of homogeneous eigensolutions is dealt using some notions from the theory of linear autonomous delay differential equations [64].

The eigenvalues of (1.20) are the roots of (1.21). It is known that, for every $\alpha \in \mathbb{R}$, (1.20) has a finite number of eigenvalues such that $\Re(\lambda) > \alpha$ (see Hale et. al. [1, Lemma 4.1]).

Let ν_0 have the meaning from Lemma 4.3.1. It will be shown that under (4.1) the eigenvalues of (1.20) with $\Re(\lambda) > -\nu_0$ coincide with the eigenvalues of M, given by (4.3) and (4.4), and the corresponding eigensolutions of (4.3) satisfy the ODE (4.2).

Proposition 4.3.4. *Let* $M \in \mathbb{R}^{n\times n}$ *be a solution of* (4.3) *with propery* (4.4). *Then* $\forall \underline{v} \in \mathbb{R}^n$, $\underline{x}(t) = e^{Mt}\underline{v}$ *is an entire solution of* (1.20).

Proof. We have $\dot{\underline{x}}(t) = M\mathrm{e}^{Mt}\underline{v}$. Since M is the solution of (4.3), this implies

$$\dot{\underline{x}}(t) = A_0\mathrm{e}^{Mt}\underline{v} + \int_{-\tau}^{0} A_\tau(\eta)\mathrm{e}^{M(t+\eta)}\underline{v}\mathrm{d}\eta = A_0\underline{x}(t) + \int_{-\tau}^{0} A_\tau(\eta)\underline{x}(t+\eta)\mathrm{d}\eta \quad \text{for } t \in \mathbb{R}.$$

□

In addition, the following lemmas are needed regarding the uniqueness of the solution with certain exponential growth at $-\infty$.

Lemma 4.3.5. *Suppose* (4.1) *holds. If* $\underline{x}_1$ *and* $\underline{x}_2$ *are entire solutions of* (1.20) *with* $\underline{x}_1(0) = \underline{x}_2(0)$ *such that*

$$\sup_{t\le 0} \|\underline{x}_l(t)\|\mathrm{e}^{\nu_0 t} < \infty, \quad l = 1, 2,$$

then $\underline{x}_1 = \underline{x}_2$ *on* $\mathbb{R}$.

Proof. Let

$$c = \sup_{t\le 0} \|\underline{x}_1(t) - \underline{x}_2(t)\|\mathrm{e}^{\nu_0 t}. \tag{4.14}$$

By (4.14), we have $0 \le c < \infty$.

For $t \le 0$ and $l = 1, 2$ we have

$$\underline{x}_l(t) = \underline{x}_l(0) - A_0\int_t^0 \underline{x}_l(s)\mathrm{d}s - \int_t^0\int_{-\tau}^0 A_\tau(\eta)\underline{x}_l(s+\eta)\mathrm{d}s\mathrm{d}\eta,$$

from (1.20). Since $\underline{x}_1(0) = \underline{x}_2(0)$, this yields for $t \in \mathbb{R}$,

$$\begin{aligned}
\|\underline{x}_1(t) - \underline{x}_2(t)\| &\le \|A_0\|\int_t^0 \|\underline{x}_1(s) - \underline{x}_2(s)\|\mathrm{d}s \\
&\quad + \int_t^0 \|A_\tau(\eta)\|\int_{-\tau}^0 \|\underline{x}_1(s+\eta) - \underline{x}_2(s+\eta)\|\mathrm{d}s\mathrm{d}\eta \\
&\le \|A_0\|c\int_t^0 \mathrm{e}^{-\nu_0 s}\mathrm{d}s + c\int_t^0 \|A_\tau(\eta)\|\mathrm{e}^{-\nu_0\eta}\mathrm{d}\eta\int_{-\tau}^0 \mathrm{e}^{-\nu_0 s}\mathrm{d}s \\
&= c\left(\|A_0\| + \int_{-\tau}^0 \|A_\tau(\eta)\|\mathrm{e}^{-\nu_0\eta}\mathrm{d}\eta\right)\int_t^0 \mathrm{e}^{-\nu_0 s}\mathrm{d}s \\
&\le c\,\underbrace{\frac{1}{\nu_0}\left(\|A_0\| + \int_{-\tau}^0 \|A_\tau(\eta)\|\mathrm{e}^{-\nu_0\eta}\mathrm{d}\eta\right)}_{\chi}\mathrm{e}^{-\nu_0 t}.
\end{aligned}$$

From the last inequality we get

$$\|\underline{x}_1(t) - \underline{x}_2(t)\|\mathrm{e}^{\nu_0 t} \le c\chi, \text{for } t \le 0,$$

which yields $c \le \chi c$, where $\chi < 1$ by (4.1). Hence $c = 0$ and $\underline{x}_1(t) = \underline{x}_2(t)$, $\forall t \le 0$. It also implies that $\underline{x}_1(t) = \underline{x}_2(t)$, $\forall t \in \mathbb{R}$, according to [1, Theorem 2.3]. □

Lemma 4.3.6. *Suppose* (4.1) *holds. Then for all* $\underline{v} \in \mathbb{R}^n$, (1.20) *has exactly one entire solution* $\underline{x}$ *with* $\underline{x}(0) = \underline{v}$ *and satisfying* (4.14) *given by*

$$\underline{x}(t) = \mathrm{e}^{Mt}\underline{v} \quad t \in \mathbb{R}, \tag{4.15}$$

where $M \in \mathbb{R}^{n\times n}$ *is the solution of* (4.3) *which satisfies* (4.4).

Proof. By Proposition 4.3.4 $\underline{x}(t)$ defined by (4.15) is an entire solution of (1.20). From (4.4) and (4.15) we get for $t \le 0$

$$\|\underline{x}(t)\| \le \mathrm{e}^{\|M\||t|}\|\underline{v}\| \le \mathrm{e}^{-\nu_0 t}\|\underline{v}\|.$$

Hence $\sup_{t\le 0}\|\underline{x}(t)\|\mathrm{e}^{\nu_0 t} \le \|\underline{v}\| < \infty$. In other words $\underline{x}$ given by (4.15) is an entire solution of (1.20) with $\underline{x}(0) = \underline{v}$, and satisfying (4.14). The uniqueness follows from Lemma 4.3.4. □

Now the main theorem of this section can be stated and proven.

Theorem 4.3.7. *Suppose* (4.1) *holds, and define*

$$\Lambda_{\nu_0} = \{\lambda \in \mathbb{C} \mid \det R(\lambda) = 0, \Re(\lambda) > -\nu_0\},$$

where $R(\lambda)$ is given by (1.21).
Let $M \in \mathbb{R}^{n\times n}$ be the unique solution of (4.3) *with propery* (4.4). *Then $\Lambda_{\nu_0} = \sigma(M)$. Moreover, $\forall\lambda \in \Lambda_{\nu_0}$* (1.20) *and* (4.2) *have the same eigensolutions corresponding to λ.*

Proof. Let $\lambda \in \Lambda_{\nu_0}$. Since $\det R(\lambda) = 0$, $\exists\underline{v} \in \mathbb{R}^n$, $\underline{v} \ne \underline{0}$, such that $R(\lambda)\underline{v} = \underline{0}$, which implies that $\underline{x}(t) = \mathrm{e}^{\lambda t}\underline{v}$ is an entire solution of (1.20). From $\Re(\lambda) > -\nu_0$ we have

$$\|\underline{x}(t)\| = |\mathrm{e}^{\lambda t}|\|\underline{v}\| = \mathrm{e}^{\Re(\lambda)t}\|\underline{v}\| \le \mathrm{e}^{-\nu_0 t}\|\underline{v}\| \quad \text{for } t \le 0,$$

which implies (4.14), i.e. $\underline{x}(t) = \mathrm{e}^{\lambda t}\underline{v}$ is an entire solution of (1.20) with $\underline{x}(0) = \underline{v}$, satisfying (4.14).

From Lemma 4.3.6 we have $\mathrm{e}^{\lambda t}\underline{v} = \mathrm{e}^{Mt}\underline{v}$, for all $t \in \mathbb{R}$. Furthermore,

$$\begin{aligned} \mathrm{e}^{\lambda t}\underline{v} - \underline{v} &= \mathrm{e}^{Mt}\underline{v} - \underline{v} \\ (\mathrm{e}^{\lambda t} - 1)\underline{v} &= (\mathrm{e}^{Mt} - I_n)\underline{v} \\ \frac{\mathrm{e}^{\lambda t} - 1}{t}\underline{v} &= \frac{\mathrm{e}^{Mt} - I_n}{t}\underline{v} \quad \text{for } t \in \mathbb{R}^*, \end{aligned}$$

we obtain $\lambda\underline{v} = M\underline{v}$ by letting $t \to 0$, which proves that $\lambda \in \sigma(M)$ and hence $\Lambda_{\nu_0} \subset \sigma(M)$.

Let $\lambda \in \sigma(M)$ and $\underline{v} \in \mathbb{R}^n$, $\underline{v} \ne \underline{0}$, such that $M\underline{v} = \lambda\underline{v}$. According to Proposition 4.3.4, $\underline{x}(t) = \mathrm{e}^{Mt}\underline{v} = \mathrm{e}^{\lambda t}\underline{v}$ is an entire solution of (1.20). Thus $R(\lambda)\underline{v} = 0$ and $\det R(\lambda) = 0$. It is known that $\rho(M) \le \|M\|$, which together with (4.4) gives

$$|\Re(\lambda)| \le |\lambda| \le \rho(M) \le \|M\| < \nu_0.$$

Therefore $\Re(\lambda) > -\nu_0$, which proves that $\sigma(M) \subset \Lambda_{\nu_0}$.

Let $\lambda \in \Lambda_{\nu_0} = \sigma(M)$. Every eigensolution corresponding to λ of (4.2) is an eigensolution of (1.20) according to Proposition 4.3.4. If $\underline{x}$ is an eigensolution of (4.2) corresponding to λ, then $\underline{x}(t) = \underline{p}(t)\mathrm{e}^{\lambda t}$, where $\underline{p}(t) \in \mathbb{R}^n$ is a polynomial in t with $\mathrm{Ord}(p(t)) = m \in \mathbb{N}^*$. There exists $k > 0$ such that

$$\|\underline{p}(t)\| \le k(1 + |t|^m), \quad \text{for all } t \in \mathbb{R}.$$

Since $\Re(\lambda) > -\nu_0$, we have $\epsilon = \Re(\lambda) + \nu_0 > 0$, so for $t \le 0$

$$\|\underline{x}(t)\| = \|\underline{p}(t)\||\mathrm{e}^{\lambda t}| \le k(1 + |t|^m)\mathrm{e}^{(\epsilon-\nu_0)t}.$$

In other words

$$\|\underline{x}(t)\|e^{\nu_0 t} \le k(1+|t|^m)e^{\epsilon t} \to 0 \quad \text{as } t \to -\infty.$$

Thus $\underline{x}$ is an entire solution of (1.20) which satisfies (4.14). Moreover, by Lemma 4.3.6, $\underline{x}(t) = e^{Mt}\underline{v}$ is also a solution of (4.2). □

According to Theorem 4.3.7, the number of eigenvalues (counting multiplicities) in Λ_{ν_0} is n, and it contains the rightmost eigenvalues of the DIDE (1.20). All the other eigenvalues of the DIDE satisfy $\Re(\lambda) \le -\nu_0$.

Proposition 4.3.8. *Suppose* (4.1) *holds. Then every* $\lambda \in \Lambda_{\nu_0} = \sigma(M)$ *is situated in the open disk* $|\lambda| < \nu_0$.

Proof. By Theorem 4.3.7, we have that $\Lambda_{\nu_0} = \sigma(M)$. Therefore if $\lambda \in \Lambda_{\nu_0} = \sigma(M)$, then $|\lambda| \le \rho(M) \le \|M\| < \nu_0$, where the last inequality follows from (4.4). □

Remark 4.3.9. We can give verifiable sufficient conditions for assumption (4.1) to hold. Taking $\nu = 1/\tau$ condition (4.1) reduces to

$$\|A_0\| + \int_{-\tau}^{0} \|A_\tau(\eta)\|e^{-\frac{\eta}{\tau}}d\eta < \frac{1}{\tau}$$

which is a generalisation of the explicit smallness condition

$$(\|A_0\| + \int_{-\tau}^{0} \|A_\tau(\eta)\|d\eta)\tau e < 1$$

used by Driver [37].

Remark 4.3.10. To evaluate the smallness condition (4.1), an integral term has to be computed. In the case where $\|A_\tau(\eta)\|$ is an elementary function, such as a polynomial function, the integral term can be computed analytically using standard calculus. In other cases, one can use the supremum of the matrix norm induced by the infinity norm, in which case the upper bound of $\|A_\tau(\eta)\|$ can be applied to check the inequality.

Example 4.3.1. *(Smallness chart)*

Let an equivalent system with (1.20) such that the maximum lag is $\overline{\tau} = 1$ and $\overline{A}_0 = \tau A_0$, $\overline{A}_\tau : [-1,0] \to \mathbb{R}^{n\times n}$, $\overline{A}_\tau(\eta) = A_\tau(\tau\eta)/\tau^2$.

Let the matrix norm be induced by the infinity norm.

As such we get the smallness condition (4.1) as the inequality with two variables

$$\nu - \|\overline{A}_0\| - \|\overline{A}_\tau\|(e^\nu - 1)/\nu > 0, \tag{4.16}$$

where $\|\overline{A}_\tau\| = \sup_{\eta\in[-1,0]} \|\overline{A}_\tau(\eta)\|$.

Figure 4.1 shows the feasible parameters $(\overline{A}_0, \overline{A}_\tau)$ for which the smallness condition holds. The investigated parameter region was $\overline{A}_0 \in [0,1]$, $\overline{A}_\tau \in [0.01, 1]$ with a step size of 0.01.

4.3.3 Asymptotic equivalence

In this section a theorem is given which shows the asymptotic equivalence of (1.20) and (4.2).

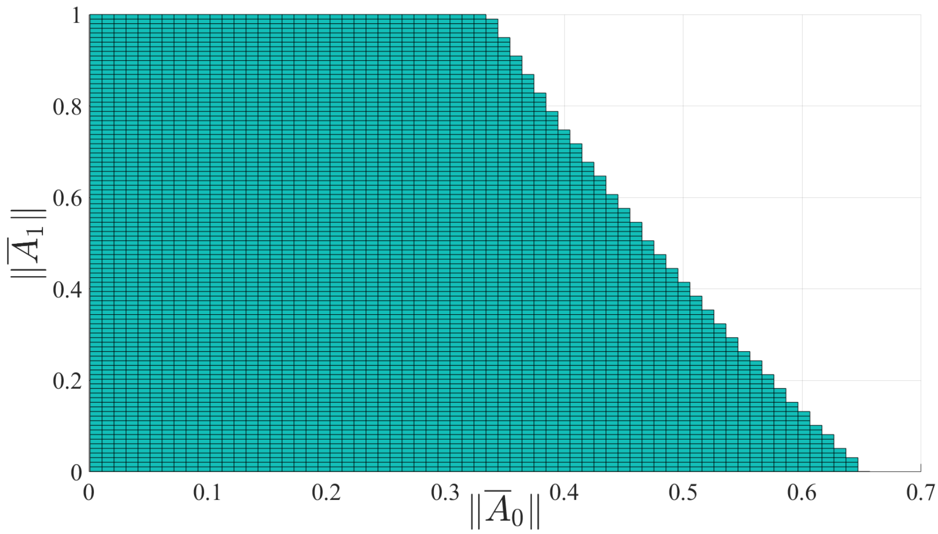

Figure 4.1 Smallness chart example for unit lag case.

Theorem 4.3.11. *Suppose* (4.1) *holds. Let* $M \in \mathbb{R}^{n\times n}$ *be the solution of the matrix equation* (4.3) *with property* (4.4). *Then the following statments are true:*

1. *Every solution of the ODE* (4.2) *is an entire solution of the DIDE* (1.20).
2. *For every solution* $\underline{x} : [-\tau, \infty) \to \mathbb{R}^n$ *of the DIDE* (1.20) *corresponding to some continuous initial function* $\underline{\theta} : [-\tau, 0] \to \mathbb{R}^n$, *there exists a solution* $\hat{\underline{x}}(t)$ *of the ODE* (4.2) *such that*

$$\underline{x}(t) = \hat{\underline{x}}(t) + \underline{o}(\mathrm{e}^{-\nu_0 t}) \quad \text{as } t \to \infty. \tag{4.17}$$

Proof. Conclusion 1 follows from Proposition 4.3.4. Conclusion 2 will be proven by applying Proposition 2.3.11 with $\gamma = -\nu_0$. It needs to be verified that the characteristic equation (1.21) has no root on the vertical line $\Re(\lambda) = \nu_0$. Suppose for contradition that there exists $\lambda \in \mathbb{C}$ such that $\det(R(\lambda)) = 0$ and $\Re(\lambda) = \nu_0$. Then there exists a nonzero vector $\underline{v} \in \mathbb{R}^n$ such that $R(\lambda)\underline{v} = \underline{0}$ and hence

$$\lambda\underline{v} = A_0\underline{v} + \int_{-\tau}^{0} A_\tau(\eta)\mathrm{e}^{\lambda\eta}\mathrm{d}\eta\underline{v}.$$

From this, we find that

$$|\lambda|\|\underline{v}\| \leq \|A_0\|\|\underline{v}\| + \int_{-\tau}^{0} \|A_\tau(\eta)\||\mathrm{e}^{\lambda\eta}|\mathrm{d}\eta\|\underline{v}\| =$$
$$= \left(\|A_0\| + \int_{-\tau}^{0} \|A_\tau(\eta)\|\mathrm{e}^{\Re(\lambda)\eta}\mathrm{d}\eta\|\right)\|\underline{v}\| = \left(\|A_0\| + \int_{-\tau}^{0} \|A_\tau(\eta)\|\mathrm{e}^{-\nu_0\eta}\mathrm{d}\eta\|\right)\|\underline{v}\| \tag{4.18}$$

Hence $|\lambda| \leq \|A_0\| + \int_{-\tau}^{0} \|A_\tau(\eta)\|\mathrm{e}^{-\nu_0\eta}\mathrm{d}\eta$, which together with Lemma 4.3.1, yields

$$\nu_0 = |\Re(\lambda)| \leq |\lambda| \leq \|A_0\| + \int_{-\tau}^{0} \|A_\tau(\eta)\|\mathrm{e}^{-\nu_0\eta}\mathrm{d}\eta < \nu_0,$$

a contradiction. Thus, Proposition 2.3.11 can be applied with $\gamma = -\nu_0$, which implies that the asymptotic relation (4.17) holds with

$$\underline{\hat{x}}(t) = \sum_{j=1}^{l} \underline{p}_j(t) e^{\lambda_j t}, \tag{4.19}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_l$ are those eigenvalues of (1.20) which have real part greater than $-\nu_0$ and $\underline{p}_j(t)$ are $\mathbb{R}^n$-valued polynomials in t. According to Theorem 4.3.7, the eigensolutions of (1.20) corresponding to the eigenvalues with real part greater than $-\nu_0$ are solution of the ODE (4.2). Hence $\underline{\hat{x}}(t)$ given by (4.19) is a solution of (4.2). □

4.3.4 Approximation of characteristic roots

According to Theorem 4.3.7, if (4.1) holds, then the dominant eigenvalues with $\Re(\lambda) > -\nu_0$ of (1.20) coincide with the eigenvalues of M, the unique solution of the matrix equation (4.3) satisfying (4.4). By Theorem 4.3.3, M can be approximated by the sequence of matrices $\{M_k\}_{k=0}^{\infty}$ defined by (4.7). As a consequence, the dominant eigenvalues of the DIDE (1.20) can be approximated by the eigenvalues of M_k. The explicit estimate (4.12) for $\|M - M_k\|$, combined with Proposition 2.3.13, yields the following result.

Theorem 4.3.12. *Suppose* (4.1) *holds so that the dominant eigenvalues of* (1.20) *coincide with the eigenvalues $\lambda_1, \ldots, \lambda_n$ of matrix M from Theorem 4.3.2. If $\{M_k\}_{k=0}^{\infty}$ is the sequence of matrices defined by* (4.7), *then the eigenvalues $\lambda_1^{[k]}, \ldots, \lambda_n^{[k]}$ of M_k can be renumbered such that*

$$\max_{1 \le j \le n} |\lambda_j - \lambda_j^{[k]}| \le 8 \cdot 4^{-\frac{1}{n}} n^{\frac{1}{n}} \nu_1 \kappa^{\frac{k}{n}}, \tag{4.20}$$

where ν_1 and κ have the meaning from Theorem 4.3.3.

Since $\kappa < 1$, the explicit error estimate (4.20) in Theorem 4.3.12 shows that under the smallness condition (4.1) the eigenvalues of M_k converge to the dominant eigenvalues of the DIDE (1.20) at an exponential rate as $k \to \infty$.

Example 4.3.2. Consider the DDE in the form of (1.20), with $n = 2$, $\tau = 1$ and system matrices

$$A_0 = \begin{pmatrix} -0.5 & 0 \\ 0 & 0.1 \end{pmatrix}, A_\tau(\eta) = \begin{pmatrix} 0 & 1.1 \\ 0.5 & 0 \end{pmatrix} (\eta + 1).$$

The characteristic equation from (1.21) has the form $\det R(\lambda) = \frac{1}{\lambda^4}(0.55 - 1.1\lambda + 0.55\lambda^2 + 0.05\lambda^4 - 0.4\lambda^5 - \lambda^6 + 1.1(-1 + \lambda)e^{-\lambda} + 0.55e^{-2\lambda})$. The values of the quantities ν_1 and κ from Theorem 4.3.12, found using *fsolve*, are $\nu_1 \approx 1.44767$ and $\kappa \approx 0.398277$. Assumption (4.1) is satisfied for any $\nu \in (1.44767, 2.96602]$. By the application of Theorems 4.3.7 and 4.3.12, it is concluded that in the region $\Re(\lambda) > -\nu_0 = -2.96602$ the characteristic equation $\det R(\lambda) = 0$ has exactly two roots λ_1 and λ_2. Furthermore, the eigenvalues $\lambda_1^{[k]}$ and $\lambda_2^{[k]}$ of the successive approximations M_k given by (2.6) can be renumbered such that

$$\max_{j=1,2} |\lambda_j - \lambda_j^{[k]}| \le \epsilon_k, \quad \text{where } \epsilon_k = 1.97064 \cdot 0.398277^{k/2}. \tag{4.21}$$

The roots of the characteristic equation $\det R(\lambda) = 0$ satisfying $\Re(\lambda) > -\nu_0$, found using *fsolve*, are $\lambda_1 = -0.772895996050269$ and $\lambda_2 = 0.254383898408625$. The approximations $\lambda_1^{[k]}$ and $\lambda_2^{[k]}$ of λ_1 and λ_2 were computed in MATLAB (see Table 4.1). The numerical results are in full agreement with the error estimate (4.21).

Table 4.1 Approximation of the characteristic roots in Example 4.3.2

k	$\lambda_1^{[k]}$	$\lambda_2^{[k]}$	$\lvert\lambda_1 - \lambda_1^{[k]}\rvert$	$\lvert\lambda_2 - \lambda_2^{[k]}\rvert$	ϵ_l
1	-0.6768	0.2770	$0.9593e-1$	$0.2259e-1$	1.9706
5	-0.7726	0.2544	$0.1019e-3$	$0.3038e-5$	0.3126
10	-0.7727	0.2544	$0.2034e-7$	$0.1418e-10$	$0.3129e-1$
15	-0.7727	0.2544	$0.3794e-11$	$0.1305e-13$	$0.3133e-2$
20	-0.7727	0.2544	$0.2998e-14$	$0.1105e-13$	$0.3136e-3$
25	-0.7727	0.2544	$0.1998e-14$	$0.1105e-13$	$0.1981e-4$

The following example shows the properties of the scalar function from 4.9 applied to Example 4.3.2.

Example 4.3.3. Consider the system from Example 4.3.2. The apropriate scalar function and its derivative are written as

$$f(\nu) = \nu - 0.5 + 1.1\frac{\nu - e^{\nu} + 1}{\nu^2}$$
$$f'(\nu) = 1 - 1.1\frac{\nu + 2 + (\nu - 2)e^{\nu}}{\nu^3}.$$

Solving $f'(\nu_0) = 0$ in the region $\nu_0 \in \mathbb{R}_+^*$ gives $\nu_0 = 2.966$. Solving $f(\nu_1) = 0$ in the interval $\nu_1 \in (0, \nu 0)$ gives $\nu_1 = 1.4477$.

Figure 4.2 shows the properties and the zero points of the scalar function and its derivative together with the maximum point of f.

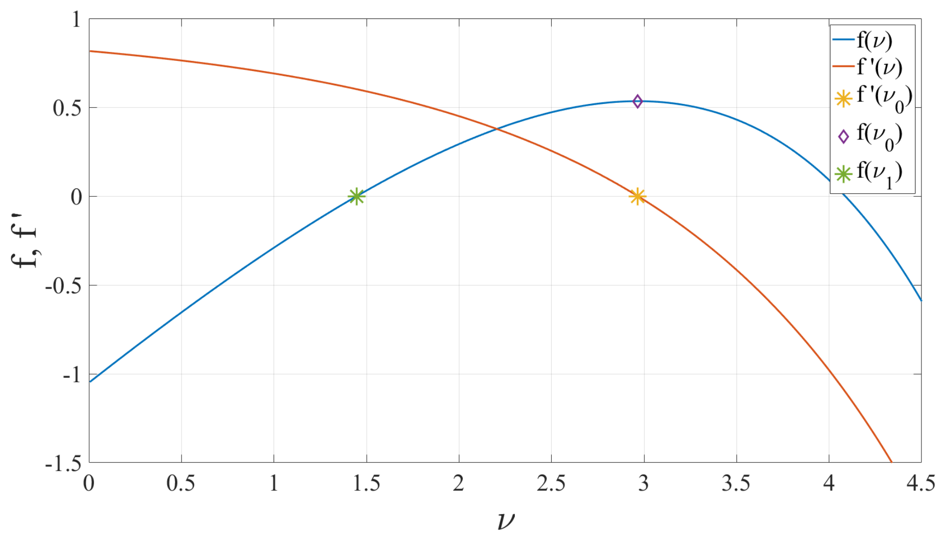

Figure 4.2 The function $f(\nu)$ and its derivative.

4.4 Extension to non-homogeneous systems

This section shows that the proposed approximation method can be extended to input affine non-homogeneous systems with distributed delay.

Theorem 4.4.1. *Consider*

$$\dot{\underline{x}}(t) = A_0\underline{x}(t) + \int_{-\tau}^{0} A_\tau(\eta)\underline{x}(t+\eta)\mathrm{d}\eta + \underline{b}(t) \tag{4.22}$$

which satisfies the smallness condition (4.1)*, and the corresponding system*

$$\dot{\hat{\underline{x}}}(t) = M\hat{\underline{x}}(t) + \hat{\underline{b}}(t), \tag{4.23}$$

where M is the solution of (4.3) *with property* (4.4) *and $\underline{b}, \hat{\underline{b}} : \mathbb{R} \to \mathbb{R}^n$ are continuous. If $\hat{\underline{b}}$ satisfies*

$$\hat{\underline{b}}(t) + \int_{-\tau}^{0} A_\tau(\eta) \int_{t+\eta}^{t} \mathrm{e}^{M(\eta+t-s)}\hat{\underline{b}}(s)\mathrm{d}s\mathrm{d}\eta = \underline{b}(t), \quad t \in \mathbb{R}, \tag{4.24}$$

then the following statements are valid:

1. *Every solution of ODE* (4.23) *is a solution of DIDE* (4.22)*.*
2. *For every solution $\underline{x} : [-\tau, \infty) \to \mathbb{R}^n$ of DIDE* (4.22) *corresponding to some initial function $\underline{\theta} : [-\tau, 0] \to \mathbb{R}^n$, there exists a solution $\hat{\underline{x}}$ of ODE* (4.23) *such that*

$$\underline{x}(t) - \hat{\underline{x}}(t) \to 0 \quad \text{as } t \to \infty.$$

Proof. It is known that $\hat{\underline{x}}_P(t)$ given by

$$\hat{\underline{x}}_P(t) = \int_{0}^{t} \mathrm{e}^{M(t-s)}\hat{\underline{b}}(s)\mathrm{d}s, \quad t \in \mathbb{R} \tag{4.25}$$

is a particular solution of the ordinary differential equation (4.23). It will be shown that if (4.24) holds, then $\hat{\underline{x}}_P(t)$ defined by (4.25) is also a particular solution of the delay equation (4.22). Indeed, by substituting (4.25) into (4.22) and using (4.23) with $\hat{\underline{x}} = \hat{\underline{x}}_P$, we obtain

$$\begin{aligned} M\int_{0}^{t} \mathrm{e}^{M(t-s)}\hat{\underline{b}}(s)\mathrm{d}s + \hat{\underline{b}}(t) = \\ = A_0 \int_{0}^{t} \mathrm{e}^{M(t-s)}\hat{\underline{b}}(s)\mathrm{d}s + \int_{-\tau}^{0} A_\tau(\eta) \int_{0}^{t+\eta} \mathrm{e}^{M(t+\eta-s)}\hat{\underline{b}}(s)\mathrm{d}s\mathrm{d}\eta + \underline{b}(t) \\ \text{for } t \in \mathbb{R}. \end{aligned}$$

By rearranging the terms and by applying (4.3), we get

$$\begin{aligned} \underbrace{\left(M - A_0 - \int_{-\tau}^{0} A_\tau(\eta)\mathrm{e}^{M\eta}\mathrm{d}\eta\right)}_{O_n} \int_{0}^{t} \mathrm{e}^{M(t-s)}\hat{\underline{b}}(s)\mathrm{d}s + \hat{\underline{b}}(t) \\ = - \int_{-\tau}^{0} A_\tau(\eta) \int_{t+\eta}^{t} \mathrm{e}^{M(\eta+t-s)}\hat{\underline{b}}(s)\mathrm{d}s\mathrm{d}\eta + \underline{b}(t), \end{aligned}$$

which is equivalent to (4.24). Thus, $\hat{\underline{x}}_P(t)$ given by (4.25) is a common particular solution of equations (4.22) and (4.23).

Now the conclusion follows from the fact that, according to Theorem 4.3.7, the corresponding homogeneous equations (1.20) and (4.2) are asymptotically equivalent.

□

Nex it will be shown that if $\underline{b} : \mathbb{R} \to \mathbb{R}^n$ is bounded, then (4.24) can be solved with respect to $\hat{\underline{b}}$ using an iterative method.

Theorem 4.4.2. *Let $\underline{b} : \mathbb{R} \to \mathbb{R}^n$ be a continuous and bounded function and suppose that* (4.1) *holds. Then* (4.24) *has a unique bounded solution $\hat{\underline{b}} : \mathbb{R} \to \mathbb{R}^n$ which can be written as*

$$\hat{\underline{b}}(t) = \lim_{k\to\infty} \hat{\underline{b}}_k(t), \quad t \in \mathbb{R},$$

where $\hat{\underline{b}}_0(t) = \underline{b}(t)$ for all $t \in \mathbb{R}$, and

$$\hat{\underline{b}}_{k+1}(t) = \underline{b}(t) - \int_{-\tau}^{0} A_\tau(\eta) \int_{t+\eta}^{t} \mathrm{e}^{M(\eta+t-s)} \hat{\underline{b}}_k(s) \mathrm{d}s\mathrm{d}\eta, \quad t \in \mathbb{R}, k = 0, 1, \ldots$$

Proof. Let $\mathcal{B} = \mathcal{BC}(\mathbb{R}, \mathbb{R}^n)$ denote the Banach space of bounded and continuous functions on $\mathbb{R}$ with the supremum norm,

$$\|\hat{\underline{b}}\|_{\mathcal{B}} = \sup_{t\in\mathbb{R}} \|\hat{\underline{b}}(t)\|, \quad \hat{\underline{b}} \in \mathcal{B}.$$

On $\mathcal{B}$, define an operator T by

$$(T\hat{\underline{b}})(t) = \underline{b}(t) - \int_{-\tau}^{0} A_\tau(\eta) \int_{t+\eta}^{t} \mathrm{e}^{M(\eta+t-s)} \hat{\underline{b}}(s) \mathrm{d}s\mathrm{d}\eta$$

for $\hat{\underline{b}} \in \mathcal{B}$ and $t \in \mathbb{R}$.

For $\hat{\underline{b}}_1, \hat{\underline{b}}_1 \in \mathcal{B}$ and $t \in \mathbb{R}$, we have

$$\begin{aligned} \|(T\hat{\underline{b}}_1)(t) - (T\hat{\underline{b}}_2)(t)\| &\le \int_{-\tau}^{0} \|A_\tau(\eta)\| \int_{t+\eta}^{t} \mathrm{e}^{\|M\|(s-t-\eta)} \|\hat{\underline{b}}_1(s) - \hat{\underline{b}}_2(s)\| \mathrm{d}s\mathrm{d}\eta = \\ &\le \|\hat{\underline{b}}_1 - \hat{\underline{b}}_2\|_{\mathcal{B}} \int_{-\tau}^{0} \|A_\tau(\eta)\| \int_{t+\eta}^{t} \mathrm{e}^{-\|M\|\eta} \mathrm{d}s\mathrm{d}\eta \le \\ &\le \|\hat{\underline{b}}_1 - \hat{\underline{b}}_2\|_{\mathcal{B}} \underbrace{\int_{-\tau}^{0} \|A_\tau(\eta)\|(-\eta)\mathrm{e}^{-\nu_1\eta}\mathrm{d}\eta}_{\kappa} = \kappa \|\hat{\underline{b}}_1 - \hat{\underline{b}}_2\|_{\mathcal{B}}. \end{aligned}$$

This shows that

$$\|(T\hat{\underline{b}}_1) - (T\hat{\underline{b}}_2)\|_{\mathcal{B}} \le \kappa \|\hat{\underline{b}}_1 - \hat{\underline{b}}_2\|_{\mathcal{B}} \quad \text{for } \hat{\underline{b}}_1, \hat{\underline{b}}_1 \in \mathcal{B},$$

with $\kappa < 1$ as in (4.13).

By Lemma 4.3.1 we have

$$\kappa = \int_{-\tau}^{0} (-\eta)\|A_\tau(\eta)\|\mathrm{e}^{-\nu\eta}\mathrm{d}\eta < 1.$$

Hence $T : \mathcal{B} \to \mathcal{B}$ is a contraction. The unique fixed point $\hat{\underline{b}}$ of T in $\mathcal{B}$, is a solution of (4.24). Moreover, the successive approximations $\hat{\underline{b}}_k$ converge to $\hat{\underline{b}}$ as $k \to \infty$ uniformly on $\mathbb{R}$ at an exponential rate. □

A simplified equivalence relation is also provided in a particular case which is essential for control applications.

Proposition 4.4.3. *Suppose* (4.1) *holds and* $\underline{b}(t) \equiv \underline{b} \in \mathbb{R}^n$ *in equation* (4.22) *is a constant vector. Assume aslo that* $\lambda = 0$ *is not an eigenvalue of* (1.20), *i.e.*

$$\det\left(A_0 + \int_{-\tau}^{0} A_\tau(\eta)\mathrm{d}\eta\right) \neq 0.$$

If $\hat{\underline{b}}(t) = \hat{\underline{b}} \in \mathbb{R}^n$ *in* (4.23) *is the constant vector given by*

$$\hat{\underline{b}} = M\left(A_0 + \int_{-\tau}^{0} A_\tau(\eta)\mathrm{d}\eta\right)^{-1} \underline{b}, \tag{4.26}$$

then Equations (4.22) *and* (4.23) *are asymptotically equivalent, i.e. Coclusions 1 and 2 of Theorem 4.4.1 hold.*

Proof. It is easy to verify that under condition (4.26) the vector $\underline{v}_e$ given by

$$\underline{v}_e = -\left(A_0 + \int_{-\tau}^{0} A_\tau(\eta)\mathrm{d}\eta\right)^{-1} \underline{b} = -M^{-1}\hat{\underline{b}}$$

is a common steady state of Equations (4.22) and (4.23). Now the conclusion follows by the same argument as in the proof of Theorem 4.4.1. □

4.5 Control of distributed delay systems

To design a controller, one must know that the system in question is controllable/stabilisable. The following part deals with the notions of stability and stabilizability of DIDE with small delays.

4.5.1 Stabilisability

Suppose that in the system (4.22) the term $\underline{b}(t)$ has the form $\underline{b}(t) = B\underline{u}(t)$, where $B \in \mathbb{R}^{n\times m}$ is the input matrix and $\underline{u}(t) \in \mathbb{R}^m$ is the control input, i.e.

$$\dot{\underline{x}}(t) = A_0\underline{x}(t) + \int_{-\tau}^{0} A_\tau(\eta)\underline{x}(t+\eta)\mathrm{d}\eta + B\underline{u}(t), \tag{4.27}$$

Its homogeneous part is given by (1.20).

Definition 4.5.1. [118, Definition 1.1] The equilibrium solution $\underline{x}_e = \underline{0}$ of (1.20) is *stable* if for any $\epsilon > 0$ $\exists\delta = \delta(\epsilon) > 0$, such that $\|\underline{x}(t)\| < \epsilon$ for any continuous initial function $\underline{\theta}$ such that $\|\theta\| < \delta$, and $t \in \mathbb{R}_+^*$.

Definition 4.5.2. System (1.20) is said to be *asymptotically stable* if it is stable and attractive, i.e. for every solution $\underline{x}$ of (1.20), $\lim_{t\to\infty} \|\underline{x}(t)\| = 0$.

Theorem 4.5.1. *[119] The DIDE* (1.20) *is asymptotically stable if and only if every characteristic root* λ *of* (1.21) *satisfies* $\Re(\lambda) < 0$.

Definition 4.5.3. A DIDE in the form of (4.27) is *asymptotically stabilisable* if there exists a continuous control input $\underline{u} : \mathbb{R}_+ \to \mathbb{R}^m$ such that the unique solution of (4.27) satisfies $\lim_{t\to\infty} \|\underline{x}(t)\| = 0$, for any continuous initial function $\underline{\theta}$.

The following theorem is a corollary of [68, Theorem 4.2].

Theorem 4.5.2. *The DIDE* (4.27) *is asymptotically stabilisable if*

$$\operatorname{rank}\left[\lambda I - A_0 - \int_{-\tau}^{0} A_\tau(\eta)\mathrm{e}^{\lambda\eta}\mathrm{d}\eta, B\right] = n \tag{4.28}$$

is fulfilled for all $\lambda \in \mathbb{C}$, with $\Re(\lambda) \geq 0$.

The following theorem is a corollary of Theorems 4.3.7 and 4.5.2.

Theorem 4.5.3. *Suppose that* (4.1) *holds. System* (1.20) *is asymptotically stable if and only if for every eigenvalue $\lambda \in \sigma(M)$, $\Re(\lambda) < 0$. Furthermore, System* (4.27) *is stabilisable if*

$$\operatorname{rank}\left[\lambda I_n - M, B\right] = n \quad \forall \lambda \in \mathbb{C},\ \textit{with}\ \Re(\lambda) \geq 0, \tag{4.29}$$

where M is the solution of (4.3) *satisfying* (4.4).

4.5.2 Stabilisation with linear state feedback

In this section, a state feedback design method is presented for the stabilisation of the addressed class of DIDEs.

Control problem 1: Consider a system in the form of (4.27) satisfying (4.1) and the reference system

$$\dot{\underline{x}}(t) = M^*\underline{x}(t), \tag{4.30}$$

where $\Re(\lambda) < 0$ for all $\lambda \in \sigma(M^*)$ and

$$\|M^*\| \leq \|M\|, \tag{4.31}$$

with M as in Theorem 4.3.2. Design a state feedback controller $\underline{u}(t) = K\underline{x}(t)$, $K \in \mathbb{R}^{m\times n}$ such that its rightmost eigenvalues match the eigenvalues of the reference system (4.30).

Theorem 4.5.4. *Consider* (4.27) *satisfying* (4.1)*. Let M^* be the state matrix of the reference system* (4.30) *such that* (4.31) *holds. If*

$$\operatorname{rank}[B, A_0^* - A_0] = \operatorname{rank}(B), \tag{4.32}$$

where

$$A_0^* = M^* - \int_{-\tau}^{0} A_\tau(\eta)\mathrm{e}^{M^*\eta}\mathrm{d}\eta \ \in \mathbb{R}^{n\times n} \tag{4.33}$$

holds and (4.27) *is stabilisable, then the control input*

$$\begin{aligned} &K = B^{+}(A_0^* - A_0),\ K \in \mathbb{R}^{m\times n} \\ &\underline{u}(t) = K\underline{x}(t). \end{aligned} \tag{4.34}$$

solves Control problem 1.

Proof. Let A_0^* given by (4.32). Since the control influences only the delay-free part of the system (4.27), the system

$$\dot{\underline{x}}(t) = A_0^*\underline{x}(t) + \int_{-\tau}^{0} A_\tau(\eta)\underline{x}(t+\eta)\mathrm{d}\eta,$$

is asymptotically equivalent to the reference system (4.30).

The controlled DIDE has the form

$$\underline{\dot{x}}(t) = (A_0 + BK)\underline{x}(t) + \int_{-\tau}^{0} A_\tau(\eta)\underline{x}(t+\eta)\mathrm{d}\eta.$$

The control gain can be computed if the matrix equation

$$A_0^* - A_0 = BK,$$

admits a solution, i.e. the rank condition (4.32) is fulfilled. In this case, K can be computed as $K = B^+(A_0^* - A_0)$ and the controlled system becomes asymptotically equivalent to the reference system.

□

4.5.3 Setpoint tracking

Control problem 2: Consider a system given by (4.27) satisfying (4.1). Let a controller $\underline{u}(t) = K\underline{x}(t) + \underline{u}_f(t)$, where K is given by (4.34) be the stabilising state feedback designed using a stable reference system as it is described in the previous section. Design the feedforward term $\underline{u}_f(t) : [-\tau, \infty) \to \mathbb{R}^n$ to ensure that the solution $\underline{x}(t)$ of (4.27) converges asymptotically to a given constant setpoint $\underline{x}_r \in \mathbb{R}^n$, i.e. $\lim_{t\to\infty} \|\underline{x}_r - \underline{x}(t)\| = 0$.

Proposition 4.5.5. *Consider the reference system*

$$\underline{\dot{x}}(t) = M^*\underline{x}(t) + \underline{b}^*, \tag{4.35}$$

where

$$\underline{b}^* = -M^*\underline{x}_r. \tag{4.36}$$

If

$$\operatorname{rank}\left[B, \left(A_0 + \int_{-\tau}^{0} A_\tau(\eta)\mathrm{d}\eta\right)(M^*)^{-1}\right] = \operatorname{rank}(B), \tag{4.37}$$

then the controller

$$K_{ff} = B^+\left(A_0 + \int_{-\tau}^{0} A_\tau(\eta)\mathrm{d}\eta\right)(M^*)^{-1} \tag{4.38}$$

$$\underline{u}(t) = K\underline{x}(t) + K_{ff}\underline{b}^* \tag{4.39}$$

solves Control problem 2.

Proof. If the control (4.39) is substituted into (4.27) we get

$$\underline{\dot{x}}(t) = A_0\underline{x}(t) + \int_{-\tau}^{0} A_\tau(\eta)\underline{x}(t+\eta)\mathrm{d}\eta + BK\underline{x}(t) + BK_{ff}\underline{b}^*.$$

From Theorem 4.5.4 it is known that $A_0 + BK = A_0^*$, which ensures the homogeneous part is asymptotically equivalent to the reference ODE (4.30). Since the reference system is asymptotically stable, it has a bounded constant steady state $\underline{x}_{ss}$ which is matched by using Proposition 4.4.3:

$$BK_{ff} = \left(A_0 + \int_{-\tau}^{0} A_1(\eta)\mathrm{d}\eta\right)(M^*)^{-1},$$

from which K_{ff} yields if the rank condition (4.37) is fulfilled. □

The summary of the control design:

- Consider the controlled system (4.27)
- Consider the reference system given by (4.35) and (4.36)
- Check the approximation condition (4.1)
- Compute M using the iteration (4.7)
- Check the control design condition (4.37)
- Compute the control gain K using (4.33) and (4.34)
- Compute the control gain K_{ff} using (4.38)
- Implement the control (4.39)

4.6 Case studies

In this section the presented theoretical result are applied to a second-order system with non-constant delay distribution.

First the ODE approximation of the DIDE system is studied and the results are compared with the existing ones in the literature. Next, the proposed approximation is applied to state feedback control design.

Let the DIDE system from Example 4.3.2 with $\underline{x}(t), \underline{u}(t) \in \mathbb{R}^2$, and initial function $\underline{\theta}(h) = -\underline{1}$, for $h \in [-1, 0]$.

The scalar function $f(\nu)$ given by (4.3.1) in this case has the form

$$f(\nu) = \nu - 0.5 - 1.1 \int_{-1}^{0} (\eta + 1) \mathrm{e}^{-\nu\eta} \mathrm{d}\eta. \tag{4.40}$$

Let $\|\cdot\| = \|\cdot\|_2$.

By solving $f(\nu_1) = 0$, and $f'(\nu_0) = 0$ we get $\nu_1 = 1.44767$, $\nu_0 = 2.96602$ so the smallness condition (4.1) is fulfilled for every $\nu \in (1.44767, 2.96602]$.

The asymptotically equivalent autonomous ODE has a state matrix norm $\|M\| < 2.96602$, so the two rightmost eigenvalues satisfy $|\lambda| < 2.96602$, see Proposition 4.3.8.

Comparison with other approximation and root-finding methods:

First, the QPmR root finder algorithm [22] was applied to get the dominant roots of the system. This method uses the characteristic Quasi-Polynomial equation of the system which in our case is:

$$\frac{1}{\lambda^4}(0.55 - 1.1\lambda + 0.55\lambda^2 + 0.05\lambda^4 - 0.4\lambda^5 - \lambda^6 + 1.1(-1+\lambda)\mathrm{e}^{-\lambda} + 0.55\mathrm{e}^{-2\lambda}) = 0.$$

This approach has found the eigenvalues 0.2544, −0.7729, −5.2863, $-6.6693 \pm 5.1378j$, $-7.6709 \pm 8.863j$ in the region $-10 \le \Re(\lambda) \le 10$ and $-10 \le \Im(\lambda) \le 10$.

Second, the Galerkin's method [112] is also applied using fifth-order Legendre polynomials shifted to the interval $[-1, 0]$, with tau incorporation (Appendix A), which produced the eigenvalues 0.2544, −0.7729, $-4.6362 \pm 7.2360j$, $-4.6568 \pm 7.0351j$, −5.2504, $-6.3196 \pm 4.0822j$, $-7.5025 \pm 3.1877j$, −8.4011. The Galerkin method also provides a linear ODE approximation system for the homogeneous part with state dimension 12.

Third, the approximation method proposed in this paper was tested. By applying the iterative method (4.7) with $M_0 = O$, after 15 iteration the state matrix

$$M_i = \begin{pmatrix} -0.5655 & 0.5501 \\ 0.3091 & 0.047 \end{pmatrix},$$

with a maximal iteration error $\|M - M_i\| < 2.9851 \cdot 10^{-6}$ is found, which was computed using (4.12). The eigenvalues of M_i are 0.2544 and -0.7729, which coincide with the rightmost eigenvalues from the QPmR algorithm, and Galerkin's method, See Fig 4.3.

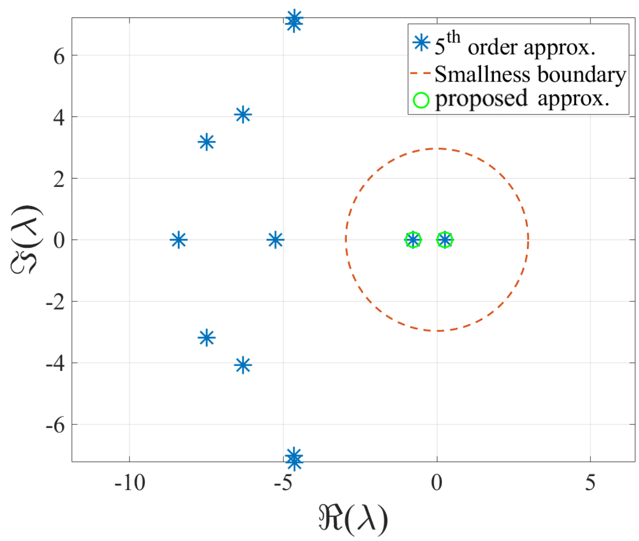

Figure 4.3 The dominant eigenvalues of the system

In Fig 4.4, the system trajectories of the 5'th order Garlekin approximation and the system trajectories of our proposed method are compared. Fig 4.5 shows the relative state error between the output of the Galerkin's approximation ($\underline{y}_g$) and the states of the approximate system $\dot{\underline{x}} = M_i \underline{x}$, i.e. $e_i = \left| \frac{x_i - \underline{y}_{gi}}{\underline{y}_{gi}} \right| \cdot 100$, $i = 1, 2$. This error has a maximum value of 1.2% in the transient region and converges to 0.25% in the steady-state region.

From these results, it can be seen that the downside of the QPmR root finder algorithm compared to the proposed method is that the region of interest is unknown apriori and must be iteratively resized to find the required number of eigenvalues. Moreover, it only gives information about the eigenvalues in contrast to our algorithm, which also provides the eigensolutions and an approximate ODE system.

The downside of the Galerkin's method is that it uses approximation systems with many state variables as an approximation of the delay system. In contrast, the proposed method gives an approximation system that preserves the number of state variables of the original system.

Control design: Let a stable reference system be

$$\dot{\underline{x}}(t) = \begin{pmatrix} -1 & 0 \\ 0 & -0.5 \end{pmatrix} \underline{x}(t) + \begin{pmatrix} 1 \\ 0.5 \end{pmatrix}.$$

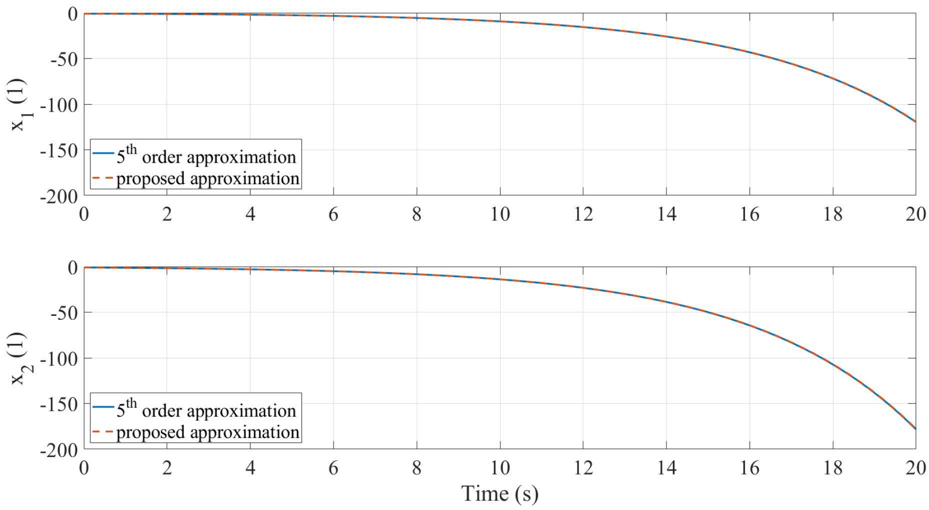

Figure 4.4 The trajectories of the approximation models

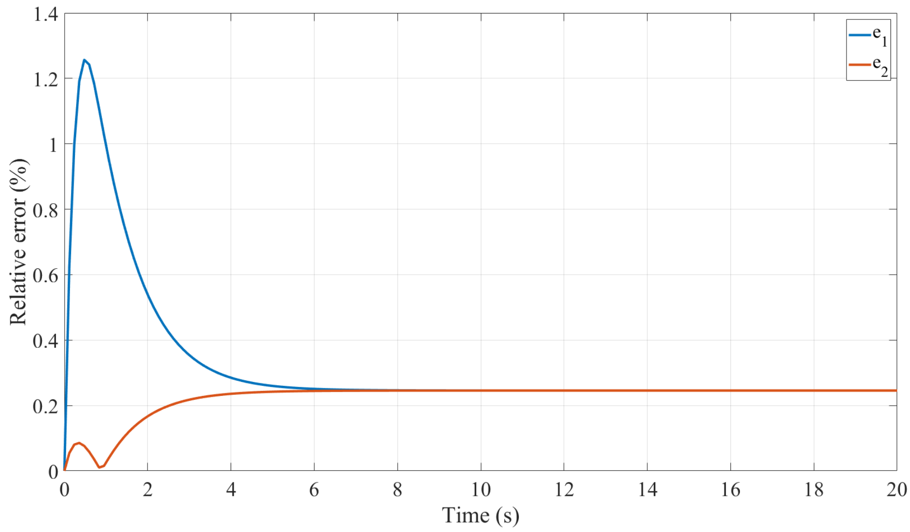

Figure 4.5 The relative state error

From (4.32) we get $A_0^* = \begin{pmatrix} -1 & -0.6544 \\ -0.3591 & -0.5 \end{pmatrix}$, and we can compute the state feedback gain as $K = \begin{pmatrix} -0.5 & -0.6544 \\ -0.3591 & -0.6 \end{pmatrix}$ according to Theorem 4.5.4. The feed-forward gain was computed according to Proposition 4.5.5 and the overall control (4.39) is

$$\underline{u}(t) = \begin{pmatrix} -0.5 & -0.6544 \\ -0.3591 & -0.6 \end{pmatrix} x(t) + \begin{pmatrix} 0.5456 \\ 0.1409 \end{pmatrix}.$$

Fig 4.6 shows that the controlled system is stable and the solution converges to the given setpoint (prescribed equilibrium point) $\underline{x}_r = \underline{1}$ with the prescribed dynamics.

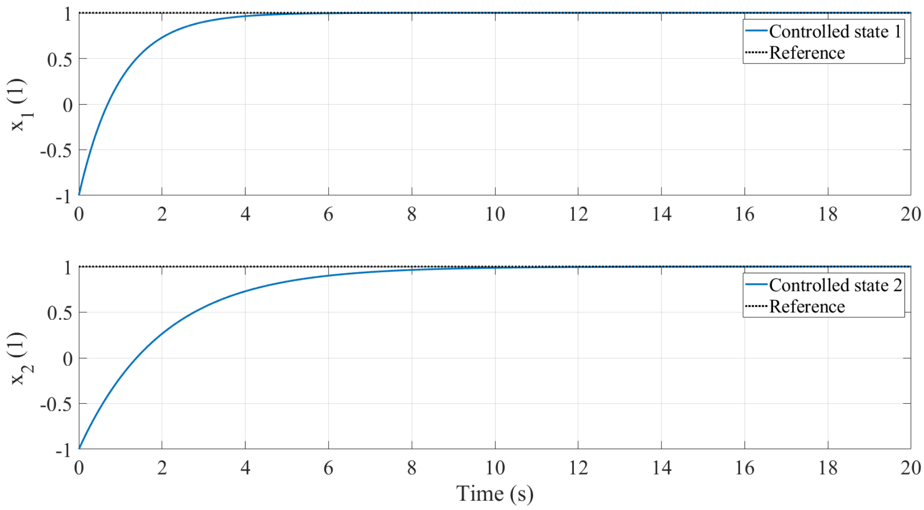

Figure 4.6 Controlled trajectories

4.7 Summary

This chapter proposes a novel approximation method for a class of linear systems with distributed delay. It has been shown that distributed delay systems can be approximated with a delay-free ODE system, which preserves the dimension of state variables if a smallness condition on the system gain holds. An exponentially convergent numerical method has been developed to compute the state matrix and implicitly the eigenvalues of the approximation system. Based on numerical computations, it was shown that the proposed approximation is in close agreement with the well-known approximation methods developed for delay systems available in the literature (QPmR, Galerkin's or spline-based methods).

The approximation method was extended to systems with a non-homogeneous term.

The approximation allows the extension of control design methods developed for delay-free systems to systems containing distributed delays. The controlled system with distributed delay will be asymptotically equivalent to the controlled delay-free approximate system used for control design. The simulation results show that the proposed control design approach can effectively be applied to unstable distributed delay systems.

Chapter 5

Conclusions and further works

This chapter aims to summarise the results of the previous chapters. Section 5.1 gives a brief review of the thesis, Section 5.2 collects the new results from the three thesis points, while Section 5.3 discusses the possible directions for future work.

5.1 Conclusions

A constructive approximation method was developed in this thesis for linear time-delay systems, which satisfy a particular smallness condition regarding the system gain and the time delay.

The algorithm finds a unique system of ordinary differential equations or ordinary difference equations for any given delay differential or delay difference system that satisfies the smallness condition. The approximating system has the same number of states as the original system containing delay.

Explicit relations were given to find the state matrices and the nonhomogeneous terms of the approximating system based on the time-delay system. Iterative relations were given with the explicit error estimates to find the solutions of the explicit relations. It has been shown that the convergence of the iterative relations is exponential. Furthermore, it has been shown that the trajectories of the approximating equations converge exponentially fast to the trajectories of the original delayed systems. Explicit estimates were given for the error between the explicit eigenvalues and the eigenvalues of the iterative solutions.

The developed approximation method was applied to continuous-time linear systems with point-wise delays, discrete Volterra-type difference equations with infinite delays and continuous-time linear systems with distributed delays.

It was shown that the proposed approximation method can be used to study the transient behaviour, the detectability and the stabilisability of systems with time delays. It was also shown that the classic observer and controller design methods, developed for classical delay free systems, are still applicable for time-delay systems using the given approximation method.

The applicability of the methods were shown through simulation use cases.

5.2 New scientific results

The new scientific results presented in this thesis are summarised in this section. They are arranged in three thesis points as follows.

Thesis 1. The asymptotic approximation of continuous-time linear homogeneous systems with point-wise delay satisfying a smallness condition was formulated. The smallness condition is represented by an inequality between the delay value and system matrix bounds. The approximation method was extended to systems with bounded, nonhomogeneous terms. The approximation method can be applied to study the detectability and to the observer design for continuous-time linear systems with point-wise delay based on classical control theoretical approaches used for linear systems without time delay.

1.1 It was shown that the addressed delay system and the approximate delay-free system are asymptotically equivalent. An iterative solution was given to find the system matrix and the nonhomogeneous term of the approximating system based on the original delayed system, and it was shown that the iteration error decreases exponentially fast.

It was shown that the eigenvalues of the approximating delay-free system coincide with the dominant eigenvalues of the original delay system, and that the eigenvalues of the approximate system converge to the eigenvalues of the original system exponentially fast.

1.2 The homogeneous system with time delay was extended with additive and bounded nonhomogeneous terms and an extended iterative approximation method was given to obtain the nonhomogeneous term of the approximate system.

1.3 It has been shown that the detectability of the addressed class of delayed system can be analysed with the help of the approximating ordinary differential equation. Furthermore, it was shown that the classical observer design methods are still valid when used in conjunction with the approximating system.

The corresponding publications are [49*], [15*], [50*], [51*], [52*] .

Thesis 2. The asymptotic approximation of discrete-time, homogeneous Volterra type difference system with infinite delays was formulated with systems of ordinary difference equations. The approximation method was extended to systems with bounded nonhomogeneous terms. The approximation method was used to study the discrete-time multi-agent systems with communication delays.

2.1 It was shown that the addressed delay system and the approximate delay-free system are asymptotically equivalent. An iterative solution was given to find the system matrix of the approximating system, and it has been shown that the iteration error decreases exponentially fast.

It was shown that the eigenvalues of the approximating delay-free system coincide with the dominant eigenvalues of the original delayed system, and that the eigenvalues of the system based on iterations converge to the dominant eigenvalues of the original system exponentially fast.

2.2 An extended approximation method was given for nonhomogeneous systems, with finite discrete delay and bounded nonhomogeneous term.

2.3 It was shown that the model of multi-agent systems that have communication delays is a particular case of the studied discrete-time Volterra type difference system. Explicit conditions were given based on the gains and the delay of the system under which the original multi-agent system with communication delay is asymptotically equivalent to a multi-agent system without delay consisting of the same number of agents.

The corresponding publications are [73*], [74*].

Thesis 3. The asymptotic approximation of continuous-time linear homogeneous systems with distributed delays that satisfy a smallness condition was formulated. The approximation method was extended to systems with bounded nonhomogeneous terms. The approximation method was used to study the stabilizability and control law design for continuous-time linear systems with distributed delay based on classical control theoretical approaches used for linear systems without delay.

3.1 It was shown that the addressed delay system and the approximate delay-free system are asymptotically equivalent. An iterative solution was given to find the system matrix and the nonhomogeneous term of the approximating system based on the original delayed system, and it was shown that the iteration error decreases exponentially fast.

It was shown that the eigenvalues of the approximating delay-free system coincide with the dominant eigenvalues of the original delay system, and that the dominant eigenvalues of the approximate system converge to the eigenvalues of the distributed delay system exponentially fast during the iterations.

3.2 An extended approximation method was given for distributed delay systems with nonhomogeneous terms.

3.3 It has been shown that the stabilizability of the original delayed system can be analysed with the help of the approximating ordinary differential equation. Furthermore, the classical control design methods are still valid when used in conjunction with the approximating system.

The corresponding publication is [97*].

5.3 Further works

Based on the presented results, the aimed future directions and applications are the following:

The basis of all three thesis points is the approximation of time-delay systems; as such, a logical next step would be to unify the approximation results using a single dynamic functional system. A possible approach to perform this is the usage of Riemann-Stieltjes or Lebesgue-Stieltjes integrals.

The observer design method was developed for the case of continuous-time systems with pointwise delays. The controller method was developed for continuous-time systems with distributed delays while the analysis of Multi-Agent Systems was done in discrete case. As a next logical step would be to extend these three methods to the other discussed system classes.

Another future direction is the extension of the studied models. In every chapter, the applicability of the proposed methods was shown in simulations using simple theoretical examples. A suitable application for the discrete-time results are the transient analysis and control of vehicle platoons or dynamically switched communication networks.

The continuous-time results for systems with distributed delays are suitable for system models in cellular biology, epidemiology and biochemistry (often time systems with small delays and small gains). This fact can be explored to implement observer and controller design framework for such systems.

Yet another application is the application of modern control methods (such as model predictive control) for delayed systems based on this approximating model.

Although the shown methods were developed for explicit computations of the dominant eigenvalues of the delayed systems and to have an asymptotic approximation of the solutions, the development of computationally efficient numerical method with explicit error estimates for solving the systems of delayed differential equations would also be a possible future work.

Appendix A

Galerkin's approximation with tau incorporation

A.1 Application to TDS with a point-wise delay

Consider the scalar system

$$\dot{x}(t) = a_0 x(t) + a_\tau x(t-\tau) + b(t) \quad \text{with } x(s) = \theta(s), \text{for all } s \in [-\tau, 0], \tag{A.1}$$

where $a_0, a_1 \in \mathbb{R}$, $b(t) : \mathbb{R}_+ \to \mathbb{R}$ is continuous and $\theta : [-\tau, 0] \to \mathbb{R}$ is a continuous initial function.

A new state variable $z(t,s) = x(t+s)$ is introduced, which yields the partial differential equation

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial s} = \frac{\partial x}{\partial (t+s)} \quad \text{for } s \in [-\tau, 0], t > 0. \tag{A.2}$$

The boundary conditions are found by substituting the new variable into (A.1),

$$\dot{z}(t,0) = a_0 z(t,0) + a_\tau z(t,-\tau) + b(t) \quad \text{with } z(0,s) = \theta(s), \text{for all } s \in [-\tau, 0], \tag{A.3}$$

where $\dot{z}(t,0) = \frac{\partial z}{\partial t}(t,0)$.

Since there is no analytical solution for the partial differential equation (A.2) with the boundary condition (A.3), the following approximation is introduced

$$z(t,s) = \underline{\phi}^\top(s)\underline{\eta}(t), \tag{A.4}$$

where $\underline{\phi} \in \mathbb{R}^p$ is the basis function (such as Legendre, Fourier or Chebyshev functions) and $\underline{\eta}(t) \in \mathbb{R}^p$ is the generalised coordinate vector.

The Legendre polinomials, shiftend to the interval $[-\tau, 0]$, were chosen as basis functions

$$\begin{aligned}
\phi_1(s) &= 1, \\
\phi_2(s) &= 1 + \frac{2s}{\tau}, \\
\phi_p(s) &= \frac{(2p-3)\phi_2(s)\phi_{p-1}(s) - (p-2)\phi_{p-2}(s)}{p-1}, \quad \text{for } p = 3, 4, \ldots.
\end{aligned}$$

Next, (A.4) is substituted into (A.2), multiply by $\underline{\phi}(s)$ and integrate over the domain $s \in [-\tau, 0]$, so

$$\begin{aligned}
\frac{\partial \underline{\phi}^\top(s)\underline{\eta}(t)}{\partial t} &= \frac{\partial \underline{\phi}^\top(s)\underline{\eta}(t)}{\partial s} \\
\underline{\phi}^\top(s)\dot{\underline{\eta}}(t) &= \underline{\phi}^\top(s)'\underline{\eta}(t) \\
\underline{\phi}(s)\underline{\phi}^\top(s)\dot{\underline{\eta}}(t) &= \underline{\phi}(s)\underline{\phi}^\top(s)'\underline{\eta}(t) \\
\underbrace{\int_{-\tau}^{0} \underline{\phi}(s)\underline{\phi}^\top(s)\mathrm{d}s}_{\Gamma}\,\dot{\underline{\eta}}(t) &= \underbrace{\int_{-\tau}^{0} \underline{\phi}(s)\underline{\phi}^\top(s)'\mathrm{d}s}_{\Psi}\,\underline{\eta}(t)
\end{aligned}$$

is obtained for $t > 0$.

Similarly, the boundary condition

$$\underline{\phi}^\top(0)\dot{\underline{\eta}}(t) = (a_0\underline{\phi}^\top(0) + a_\tau\underline{\phi}^\top(-\tau))\underline{\eta}(t) + b(t), \quad \text{for } t > 0,$$

and the initial value

$$\begin{aligned}
\underline{\phi}^\top(s)\underline{\eta}(0) &= \theta(s) \\
\underline{\phi}(s)\underline{\phi}^\top(s)\underline{\eta}(0) &= \underline{\phi}(s)\theta(s) \\
\int_{-\tau}^{0} \underline{\phi}(s)\underline{\phi}^\top(s)\mathrm{d}s\underline{\eta}(0) &= \int_{-\tau}^{0} \underline{\phi}(s)\theta(s)\mathrm{d}s \\
\underline{\eta}(0) &= \Gamma^{-1}\int_{-\tau}^{0} \underline{\phi}(s)\theta(s)\mathrm{d}s
\end{aligned}$$

is found.

In summary, the problem of solving the TDS (A.1) is transformed into the problem of solving the ODE

$$\begin{pmatrix} \Gamma \\ \underline{\phi}^\top(0) \end{pmatrix} \dot{\underline{\eta}}(t) = \begin{pmatrix} \Psi \\ (a_0\underline{\phi}^\top(0) + a_\tau\underline{\phi}^\top(-\tau)) \end{pmatrix} \underline{\eta}(t) + \begin{pmatrix} \underline{0}_p \\ \underline{b}(t) \end{pmatrix}, \tag{A.5}$$

with initial condition $\underline{\eta}(0) = \Gamma^{-1}\int_{-\tau}^{0}\underline{\phi}(s)\theta(s)\mathrm{d}s$. The system (A.5) provides $p+1$ independent equations for p variables, it is an over determined system. The least-squares solution can be computed using

$$\dot{\underline{\eta}}(t) = \begin{pmatrix} \Gamma \\ \underline{\phi}^\top(0) \end{pmatrix}^{\dagger} \left(\begin{pmatrix} \Psi \\ (a_0\underline{\phi}^\top(0) + a_\tau\underline{\phi}^\top(-\tau)) \end{pmatrix} \underline{\eta}(t) + \begin{pmatrix} \underline{0}_p \\ \underline{b}(t) \end{pmatrix} \right)$$

It was found that using the spectral tau or spectral least-squares method gives a better performance when computing the approximate solution of (A.1) [27]. The newly created ODE has the form

$$\dot{\underline{\eta}}(t) = \begin{pmatrix} \overline{\Gamma} \\ \underline{\phi}^\top(0) \end{pmatrix}^{-1} \left(\begin{pmatrix} \overline{\Psi} \\ a_0\underline{\phi}^\top(0) + a_\tau\underline{\phi}^\top(-\tau) \end{pmatrix} \underline{\eta}(t) + \begin{pmatrix} \underline{0}_{p-1} \\ \underline{b}(t) \end{pmatrix} \right),$$

which is system with p independent equations for p variables. Here $\overline{\Gamma}$ is the Γ matrix with the last row removed, and $\overline{\Psi}$ is the Ψ matrix with the last row removed.

For TDS with more than one state variable, a similar method applies.

A.2 Extension to TDS with distributed delay

Consider the scalar system

$$\dot{x}(t) = a_0 x(t) + \int_{-\tau}^{0} a_\tau(r) x(t+r)\mathrm{d}r + b(t) \quad \text{with } x(s) = \theta(s), \text{for all } s \in [-\tau, 0], \tag{A.6}$$

where $a_0 \in \mathbb{R}$, $a_\tau : [-\tau, 0] \to \mathbb{R}$ is continuous and not identically zero, $b(t) : \mathbb{R}_+ \to \mathbb{R}$ and $\theta : [-\tau, 0] \to \mathbb{R}$ are continuous. The results from Section A.1 hold for the approximating system

$$\underline{\dot{\eta}}(t) = \begin{pmatrix} \overline{\Gamma} \\ \underline{\phi}^\top(0) \end{pmatrix}^{-1} \left(\begin{pmatrix} \overline{\Psi} \\ a_0 \underline{\phi}^\top(0) + \int_{-\tau}^{0} a_\tau(r) \underline{\phi}^\top(r)\mathrm{d}r \end{pmatrix} \underline{\eta}(t) + \begin{pmatrix} \underline{0}_{p-1} \\ \underline{b}(t) \end{pmatrix} \right),$$

with initial condition $\underline{\eta}(0) = \Gamma^{-1} \int_{-\tau}^{0} \underline{\phi}(s)\theta(s)\mathrm{d}s$.

When the Galerkin's method is used, some symbolic computations are needed. MATLABs Symbolic Toolbox provides a complete environment for symbolic computations and manipulations. The *sym* expression creates the variables, *diff* and *int* can be used to compute symbolic differentiation and integration respectively. Expressions can be simplified and factored out using *simplify*. Substituting variables with other symbolic expression is done with the help of *subs*. The *double* function was used to convert symbolic variables into numerical ones.

Appendix B

Applied methods

B.1 Theoretical methods

In order to study the properties of some functions, methods from mathematical analysis were used [120]. The norm-based inequalities were proposed and proven using basic notions from metric spaces [121].

For every explicit equation regarding the state matrix and the nonhomogeneous term of the approximate system, Banach's fixed-point theorem was used combined with the notion of contraction mapping.

B.1.1 Contraction mapping

Let $(\mathcal{M}, d_{\mathcal{M}})$, $(\mathcal{N}, d_{\mathcal{N}})$ be metric spaces and $f : \mathcal{M} \to \mathcal{N}$. $f : \mathcal{M} \to \mathcal{N}$ is a *contractive mapping* if

$$d_{\mathcal{N}}(f(x), f(y)) \leq c d_{\mathcal{M}}(x, y)$$

holds $\forall x, y \in \mathcal{M}$, with a Lipschitz constant $c \in [0, 1)$.

B.1.2 Banach's fixed point theorem

Let $(\mathcal{X}, d)$ be a non-empty, complete metric space with a contraction mapping $f : \mathcal{X} \to \mathcal{X}$. Then f admits a unique fixed point $x^* \in \mathcal{X}$ such that $f(x^*) = x^*$. Furthermore, the fixed point x^* can be written as a limit of the iterations $x_k = f(x_{k-1})$ for $k \geq 1$ starting from an arbitrary element $x_0 \in \mathcal{X}$.

B.2 System theoretical and control methods

The applications of this thesis were driven by the following system theoretical and control methods: The stability of linear systems was analysed based on the location of their poles in the s-Plane [12].

The design of an asymptotically stable state observer and stabilising state feedback control was proposed using the well-known pole placement method [122].

B.3 Numerical methods for solving ODE and DDE:

Many differential equations cannot be solved analytically. For practical purposes numerical approximations of the solution are often time sufficient. Stiff differential equations are differential equations for which certain numerical methods result in numerically unstable solutions unless the integration step size is taken to be extremly small. These numerical methods fall into two major categories: Runge-Kutta methods and linear multistep methods [123]. Furthermore, there are implicit methods, for example, Adams-Moulton methods, backwards differentiation methods, diagonally implicit Runge-Kutta methods, Gauss-Radau methods, and explicit methods, for example the Adams-Bashforth methods and the Runge-Kutta methods with lower diagonal Butcher tableau [124].

A numerical method starts the solution from an initial point and tries to approximate the solution by taking short steps to find the next solution point. Single step methods (i.e. Euler's method) use only one previous point and its derivative. Runge-Kutta methods take intermediate steps to have a higher order, but all previous information is discarded before the next step. Multistep methods gain efficiency by keeping and using information from several previous steps.

In this work, the numerical solution of systems of ODEs was found with the help of the *ode23* MATLAB function, which is an implementation of the explicit Runge-Kutta (2,3) pair of Bogacki and Shampine for nonstiff differential equations [125, 126].

The method of steps is a well-known technique for the study of DDE which reduces them to a sequence of ODE. The numerical solution of DDE require more elaborate algorithms, which take into account the initial function, the discontinuity that propagates throughout the interval of interest. Because of this propagation, multiple delays cause special difficulties in the solution. Moreover, delays can vanish, the solution of such a DDE may or may not extend beyond the singular point, the solution may not be unique [127].

In this thesis the MATLAB function *dde23* was used to numerically solve the systems of DDEs with constant delays. *dde23* tracks discontinuities, uses the explicit Runge-Kutta (2,3) pair algorithm for integration and the interpolant of *ode23*. It also uses iterations for steps longer that the lags [128].

The eigenvalues of ODEs were found using the *eig* function of MATLAB, while in the case of DDEs the *QPmR* algorithm was used in a given region of interest.

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